Problem Set IV - 1986

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, student assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.
In Mathematics, the same exam serves as the GED and the entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Write the equation of the circle that contains the points $A(1,4)$ and $B(5,0)$ and whose center is on the line $x+y=3$.
2. Solve the equation over the real numbers:
$$\sqrt{3-x}+\frac{6}{\sqrt{3-x}}=\sqrt{9-5x}$$
3. Solve the equation over the real numbers.
$$2\log(x+3)-\log x=\log(25x+3)-1$$
4. Consider triangle ABC. The altitude (or height) belonging to side AB splits the side into line segments of lengths 8 and 3 units long. The angle of the triangle near the 3 unit long segment is twice the angle near the 8 unit long segment. Find the triangle's sides and area.
5. A square based straight pyramid was cut into two by a plane parallel to its base and containing the midpoint of its main altitude. The volume of the pyramid created is $336\,\text{cm}^3$, and its height is one third of the side of the base of the original pyramid. Find the other part's volume. (That is the part that is not a pyramid, also called a frustum.) Find the angle that is formed between the side face and the base of the pyramid.
6. The first term of an arithmetic sequence equals to the common ratio of a geometric sequence. The first term of this geometric sequence is the difference of the arithmetic sequence. The sum of the first five terms of the arithmetic sequence is 40, the sum of the first two terms of the geometric sequence is 10. Find these sequences.
7. Solve the following equation over the real numbers.
$$\frac{1+\tan x}{1-\tan x}=1+\sin 2x$$
8. (In Hungary at the time, the lottery rules were: we selected five out of the set $\{1,2,3,...90\}$.) A person wants to select the numbers to be played on the lottery as follows: after selecting the smallest two numbers, the third number would be the sum of the first two; the fourth number would be the sum of the first three numbers, and the fifth number would be the sum of the first four.
a) What is the greatest possible value of the smallest number in the set of five?
b) If we select the smallest number to be the greatest possible, then what numbers would be playing?
c) In how many different ways can the numbers be played as described?

