Problem Set 1

1. Simplify:

$$E = \frac{1-\left(\frac{a}{b}\right)^2}{1-\frac{a}{b}} + \frac{1-\left(\frac{b}{a}\right)^{-1}}{1-\frac{b}{a}}$$

2. A solution contains 4% salt. How much water do we need to add to the 30 milliliters of this solution so that the resulting solution will have a salt concentration of 1.5%?

3. Find an equation for the straight line that contains the origin and the length of the line segment inside the circle

$$x^2 + y^2 - 2x + 2y - 2 = 0$$

is the smallest possible.

4. Let $ABCD$ be a square. Suppose that P is the midpoint of side CD. We draw a line that is perpendicular to AP and contains P. This line intersects side BC in point Q. We draw a line that is perpendicular to PQ and contains Q. This line intersects side AB in point R. We draw a line that is perpendicular to QR and contains R. This line intersects the line segment AP in point S. Compute the ratio between the area of the quadrilateral $PQRS$ and the square $ABCD$.

5. Compute the sum of all positive two-digit integers that give you a remainder of 1 when divided by 4.

6. Solve the following system of equations:

$$\begin{aligned} \log_{xy}(x-y) &= 1 \\ \log_{xy}(x+y) &= 0 \end{aligned}$$

7. Find all x,y real numbers for which

$$x + \frac{1}{x} = 2\cos y$$

8. Let O be the vertex of an angle. On one of the rays, we select points A and B. On the other ray, we select points A' and B' so that line segment AB and $A'B'$ have equal length. The order of the points are OAB and $OA'B'$. Let $F_1 be$ the midpoint of line segment AA' and let F_2 be the midpoint of line segment BB'. Prove that the line connecting F_1 and F_2 is parallel to the angle bisector.

Problem Set 2

9. a) Define the logarithm of b to base a.

   b) What is the result if we raise 196 to the power of $\cos\left(\frac{1972\pi}{3}\right)$?

   c) Arrange the following in an increasing order:

   $$\sqrt[3]{3},\ \log_{\sqrt{2}}\left(\frac{1}{2}\right),\ 2^{\sqrt{2}},\ \sqrt[4]{4}$$

   d) What is the domain of the function $f(x) = \tan\left(\frac{\pi}{4} - x\right)$? Sketch the graph if this function.

10. In a factory, three workers take on the production of a cretain number of items. The first worker could alone complete the job in 80 days. The second and third workers could each alone complete the same job in 96 days.

    a) How many days will the job take if all three workers work together?

    b) The company awards the job with a \$2000 bonus. If the three workers are to distribute it in the proportion of their share of the work, how much should each get?

11. Every year, Robert deposits \$2000 into his savings account with an annual compound interest rate of 5%. How long until he has \$100 000 in his account?

12. The solutions of the equation $x^2+ax+b=0$ are 1 greater than the solutions of the equation $x^2+bx+a=0$. Find these solutions.

13. Find all integer values of m if we know that both solutions of the system

$$\begin{cases} x+3y=12 \\ 21x+62y=m \end{cases}$$

are non-negative?

14. One side of a square is P_1P_2 where $P_1\,(1,3)$ and $P_2\,(5,1)$. Consider the line connecting these points. The point on this line with $y-$coordinate 2 is the center of a circle with radius $\sqrt{10}$. Compute the area of the intersection of the square and the circle.

15. Consider a semicircle with radius r. We draw a recatangle into the semicircle so that one of its sides lies on the diameter and the other two vertices lie on the arc of the semicircle. Find the length of the sides of such rectangle with the greatest area.

Problem Set 3

16. The angles along a parallel side of a trapezoid are both 30°. The other three sides are 12 units long. Compute the area of the trapezoid.

17. Solve each of the following equations.

    a) $x^2=\sqrt{x}$ b) $2^x=-3$ c) $\tan 2x=1$ d) $\log_x 2x=2$

18. Solve the following equation:

$$\frac{21}{\sqrt{2x+1}}-2\sqrt{x}-\sqrt{2x+1}=0$$

19. $ABCD$ is a rectangle. Side AB lies on the line $y=3x$. the diagonals of the rectangle intersect each other at $M\,(12,6)$. Diagonal AC is parallel to the x axis. Find the coordinates of points B, C, and D.

20. Suppose that α and β are planes perpendicular to each other. Let A be any point of plane α and B be any point of plane β. Let m be the line where the planes intersect each other. Let A' and B' be the perpendicular projections of pont A and B to line m, correspondingly. Suppose that F is the midpoint of line segment AB. Prove that $FA'=FB'$.

21. The angles of a triangle are α, β, and γ. prove that if

$$2\cos\alpha=\frac{\sin\gamma}{\sin\beta}$$

then the triangle is isosceles.

22. For ten years, at the beginning of each year, a company introduces a new machine, worth A dollars. The value of each machine decreases by $p\%$ each year. What is the total worth of all ten machines at the end of the tenth year?

22. Suppose that x, y, and z are real solutions of the system

$$\begin{aligned} x+y+z &= v \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z} &= \frac{1}{v} \end{aligned}$$

Prove that one of x, y, and z must be equal to v.

Problem Set 4

24. What is the domain of a function?

25. Write the given quantities in increasing order by their value

$$\log_{1/3} 9, \quad \sin\left(\frac{17\pi}{4}\right), \quad \log_9\left(\frac{1}{3}\right)$$

26. How far is the origin from the line $x - y + 2^{10} = 0$?

27. One side of a triangle is 2 cm long, another side is 8 cm. The length of the third side in the triangle is an integer, divisible by 3. How long is this third side?

28. Graph the function $y = \dfrac{x^3-1}{x^2}$

29. A circle with radius 1 is tangent to both rays forming a right angles. Find the radius of the circle that is tangent to both rays and the circle.

30. Company A found that in the first four months of this year, it had produced exactly as planned. In the second four months of the year it produced 21% above the plan, and in the last four months of the year, it overproduced by 44%. Company B produced 69% over its plan in the first six months and overproduced by 96% in the second six months of the year. As it turns out, the companies produced exactly the same number of items. Which company's plan called for more production and by what percentage?

31. Re-write the expression $|3-x|+x$ without the use of the absolute value symbol.

32. Compute the area of the triangle bounded by the lines given:

$$y - 10^{-9}x = 10^{-6}, \quad y = 10^{-6}, \quad 2y = 10^{-9}x + 10^{-6}$$

33. Prove that if a triangle has sides $a = n^2+3n+3$, $b = n^2+2n$, $c = 2n+3$ where $n > 1$ is a natural number, then one of the angles in the triangle has measure $120°$.

Problem Set 5

34. a) Solve the given equation:

$$1+\frac{1}{3-\frac{1}{x}}=2$$

Find the exact value of each of the following:

b) $B=\left(x^2-1\right)\left(\frac{1}{x-1}-\frac{1}{x+1}-1\right)$ if $x=\sqrt{3}$

c) $C=\frac{p^2+pq+q^2}{p^2-pq+q^2}$ if $p=\frac{1}{\sqrt{2}}$ and $q=\frac{1}{\sqrt{8}}$

35. The population of a town is currently 48 300.

a) How many people lived there a year ago if during last year, there was a 5% increase in the peopulation?

b) How many people will live in the town a year from today if there will be next an 8% increase in the population?

36. Compute the value of the given expression. Is it true that its value is independent from the value of x?

$$E=\cos^2 x+\cos^2\left(30^\circ+x\right)-2\cos 30^\circ\cos x\cos\left(30^\circ+x\right)$$

37. Prove that the sum of the legs in a right triangle are equal to the sum of the diameters of the inscribed and superscribed triangles.

38. A shipping company determined that a certain shipping job could be completer if 30 smaller trucks work for 8 hours and 9 big trucks work for 6 hours. If the same large trucks work for 8 hours and the same smaller ones for 6 hours, then they would only complete $\frac{13}{15}$ of the job. How many hors needed to complete this shipping job if we only use

a) one smaller truck

b) one larger truck?

39. Solve the given equation:

$$\left(\frac{x}{3}\right)^{3+\log x}=3\cdot 10^4$$

40. Suppose that C is a circle and p and q are two perpendicular chords in C. p and q split each other into four line segments. Draw a circle around each line segment as a diameter. Prove that the sum of the four circle's area is the same as the area of the original circle.

41. Suppose that $\overline{aa...a}$ denotes an $n-$digit number. Compute the sum

$$\overline{a}+\overline{aa}+\overline{aaa}+...+\overline{aa...a}$$

where the last number has n digits.

Problem Set 6

42. Prove the following identity.

$$\frac{1+(a+x)^{-1}}{1-(a+x)^{-1}} \cdot \left[1-\frac{1-\left(a^2+x^2\right)}{2ax}\right] = \frac{(a+x+1)^2}{2ax}$$

43. Two tractors with different capacities, are working on a field. The two machines, working together would finish the work in 4 days. If one machine first completes the work on two-thirds of the field, and then the other machine completes the rest, that would take 8 days. How long would it take for each machine to complete the work on its own?

44. What is the distance between the point $P(-3,2)$ and the line $4x-3y=7$?

45. Solve the given system of equations.

$$\begin{aligned} 4^{x-y} - 4^{(x-y)/2} &= 2 \\ 5^{\log(2y-x-1)} &= 1 \end{aligned}$$

46. Solve the equation

$$\cot x + \frac{\sin x}{1+\cos x} = 2$$

47. Compute the volume of the object we obtain when we rotate the line $y=2x+3$ between $x=2$ and $x=5$ about the $x-$axis.

Problem Set 7

48. In a certain mixture of grains, the amount of corn, semolina, lucerne, rye, and barney are in a ratio of $40:15:10:12:18$ and 5% of other grains. How much of each of the grains is needed if we want to make a mixture of 720 pounds?

49. In March of this year, the average daily tempreture was below $53F$ for 20 days, and was above $47F$ for 24 days. On how many days was the daily average temperature between $47F$ and $53F$?

50. Solve the given equation:

$$\frac{6}{x-2} - \frac{26}{10-5x} = \frac{24}{2x-4} - \frac{1}{10}$$

51. The sides of a triangle are 15 cm, 20 cm, and 7 cm long. Let C be the circle that contains all three vertices of the triangle. Find the area of the sector of the circle that is being cut off by the 15 cm side.

52. A lake is in the shape of a rectangle, with sides 120 m and 160 m long. At a point on its diagonal, a pile was placed. So happens, if we connect the ends of the shorter side with the pile, the line segments are perpendicular to each other. How far is the pile from the sides of the lake?

Problem Set 8

53. Determine the exact value of $2x^2 - 3xy + 5y^2$ if given that $x = 0.5\sqrt{10} - 2\sqrt{2}$ and $y = 0.5\sqrt{10} + 2\sqrt{2}$.

54. Solve the equation

$$\sin 2x + \cos x = 0$$

55. Solve the equation over the real numbers and graph its solutions as a function of the parameter α.

$$y^2 + 2y + \cos^2 \alpha = 0$$

56. Solve the following equatin over the real numbers:

$$4 \cdot 5^{2x} - 3 \cdot 5^{x+1} - 25 = 0$$

57. Determine the distance between the center of the circle $x^2 + y^2 - 14x + 4y + 44 = 0$ and the line $y = 2x$.

58. Suppose that a circle is given with radius r. We draw a diameter of the circle. Suppose that P is a point on the circle that is at a distance of $\dfrac{2r}{3}$ from the diameter. Find the distance of P from the two endpoints of the diameter.