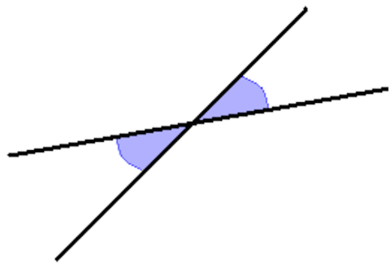
Part 1 – The Sum of the Angles in a Triangle

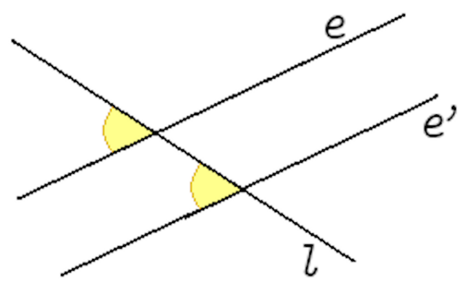
Theorem: The sum of the inner angles in any triangle is 180°.

Proof: We will use the following facts.

When two lines intersect each other, the opposite angles (as shown) formed between them have equal measure. We say that these angles are **vertical angle**s.



When a third line intersects a pair of parallel lines, the **corresponding angles** (as shown) have equal measure. This third line is called the transversal for the pair of parallel lines.



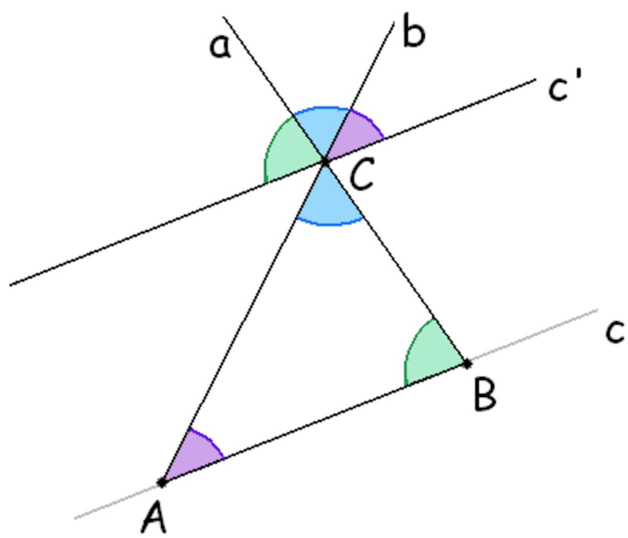
We are now ready to pove our claim. Let ABC be any triangle. We extend lines a and b beyond point C We also draw a line through point C that is parallel to side AB

The purple angles at point A and point C are equal because line b is a transversal for the parallel lines c and c' .

Similarly, the green angles at points B and C are eqiual because line a is also a transversal for the parallel lines c and c'. Finally, the two blue angles at C are equal because they are vertical angles.

To complete our proof, we need to observe two things.

First, the sum of the three angles in the triangle is the same as the sum of the other three triangles outside of the triangle, above point C. Second, those three angles add up to 180° because next to each other, they form a straight angle, 180°.



This completes our proof.

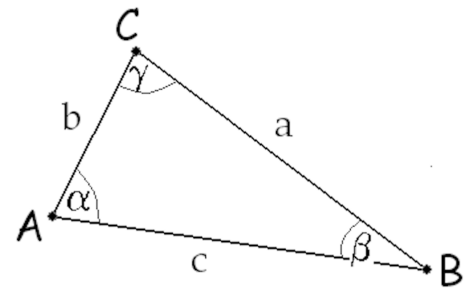
Part 2 – Standard Labeling

Standard labeling is a method of notation that automatically associates line segments, points and angles in a triangle. Standard labeling helps us avoiding confusion or lengthy explanations in geometry problems. This agreement is called *standard labeling*, and it establishes a connection between the labels of sides, vertices, and angles in triangles. Every triangle has three of the following three components.

vertices :(singular: vertex) Points are usually denoted by uppercase letters. In case of triangles, we often use A, B, and C.

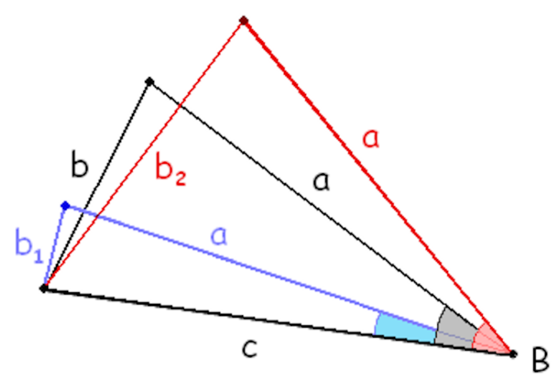
angles: Angles are usually denoted by lowercase Greek letters. In case of triangles, we often use α (alpha), β (beta), and γ (gamma).

sides: Lines and line segments are usually denoted by lowercase letters. In case of triangles, we often use a, b, and c.



In case of standard labeling, we automatically associate sides, vertices, and angles. A vertex is associated with the angle located at that vertex. These two are associated with the side opposite these. For example, angle α is always assumed to be located at point A, and side a is always assumed to be the side oppsite to point A and angle α. Point B, angle β, and side b are similarly grouped. Unless otherwise indicated, we should always assume standard labeling when presented with data that uses these letters.

Standard labeling is a smart approach to triangles, because there is a natural connection between an angle in a triangle and the side opposite that angle. Consider, for example, the triangle shown above with standard labeling. What if we fixed sides a and c and only modified angle β? Imagine that we have two rods in the lengths of a and c attached to each other at one end and we can freely change the angle between them. If we increase the angle between sides a and c (see the red lines), the side opposite will also increase. If we decrease the angle between sides a and c (see the blue lines), the side opposite will also decrease. So, there seems to be a natural correspondance between side b and angle β.



> **Theorem:** In any triangle ABC, there is a correspondance between the length of a side and the measure of the angle opposite that side: The longest side is opposite the greatest angle, and vica versa: the greatest angle is opposite the longest side. The shortest side is opposite the smallest angle, and vica versa: the smallest angle is opposite the shortest side.

So, the order between the three sides is the same as the order between the corresponding angles, and vica versa. We recommend that sides in triangles are tracked by their corresponding sides. This is because we can perceive the difference in angles much better than in side lengths.

Example 1. Suppose that ABC is a triangle with $\alpha = 82°$ and $\gamma = 39°$. List the length of the sides of the triangle in an increasing order.

Solution: Recall that the three angles in a triangle add up to $180°$. This means that if two angles are given, we can compute the third one. $\beta = 180° - (82° + 39°) = 180° - 121° = 59°$. Now we can see the order between the angles. γ is the smallest angle, β is in the middle, and α is the greatest angle. In short: $\gamma < \beta < \alpha$. The order between the lengths of the sides is the same: c is the shortest side, b is in the middle, and a is the longest side. In short: $c < b < a$.

Part 3 – Chasing Angles

There is an easy but important consequence of the connections between an angle in a triangle and the side opposite.

> **Theorem:** In any triangle ABC, if two angles have equal measures, then the sides opposite them have equal length. Conversely, if two sides are equally long, then the angles opposite those sides have equal mesaures. Such a triangle is called **isosceles**.

We will see that isosceles triangles are very commonly occuring objects in geometry problems. In many cases, we prove that two angles are equal to each other and the conclude that the sides opposite these angles have equal length. Or vica versa, if we prove that two sides are equally long, we can then conclude that the opposite angles are also equal.

> **Theorem:** In any triangle ABC, if all three angles have equal measures, then all three sides have equal length. Conversely, if three sides in a triangle are equally long, then all angles have equal mesaures. Such a triangle is called **regualar** or **equilateral or equiangular.**.

The angles in a regular traingle measure $60°$. If the three angles add up to $180°$ and are equal, $3x = 180 \implies x = 60°$.

Example 2. Suppose that one angle in an isosceles triangle is $40°$. Find the measure of the other angles in the triangle.

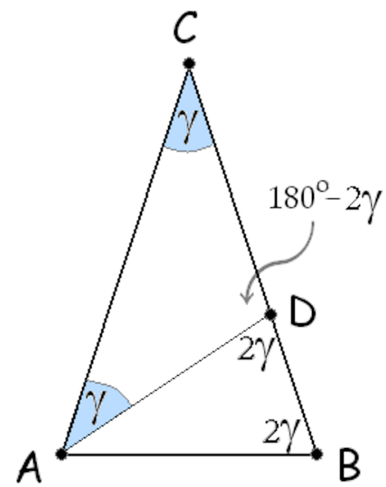
Solution: As it is phrased, this problem has several solutions. If the other angles are different from $40°$, they must be equal to each other. Therefore, we set up and solve the equation $40 + 2x = 180°$. if there is another $40°$ angle in the triangle, we have $2 \cdot 40 + x = 180$.

$$\begin{aligned} 2x + 40 &= 180 \\ 2x &= 140 \\ x &= 70 \end{aligned} \qquad\qquad \begin{aligned} x + 2 \cdot 40 &= 180 \\ x + 80 &= 180 \\ x &= 100 \end{aligned}$$

So the angles in the triangle are either $40°$, $70°$, $70°$, or $40°$, $40°$, and $100°$.

Example 3. Consider triangle ABC that is isosceles with $AC = BC$. Suppose that D is a point on side BC such that $AB = AD = DC$. Find the angles in triangle ABC.

Solution: Let γ denote the angle by C Because $AD = CD$, the angles opposite are also the same. Therefore, angle CAD is also γ. If we denote the third angle in triangle ADC by θ, then $\theta + 2\gamma = 180°$ means that $\theta = 180° - 2\gamma$. We often will skip the equation, and write $180° - 2\gamma$. We can think of this angle, as the angle, to which we add 2γ, we get $180°$. Consider now angle ADB. That too, when added to angle ADC, will result in a sum of $180°$. Therefore, angle ADB is also 2γ.



Triangle ADB is also isosceles, with $AB = AD$. Therefore, the angle at B is also 2γ. Consider now the angles in triangle ABD. Angle BDA is one that if added to 4γ, the sum is $180°$. With or without equation, we get that angle BDA is $180° - 2\gamma$.

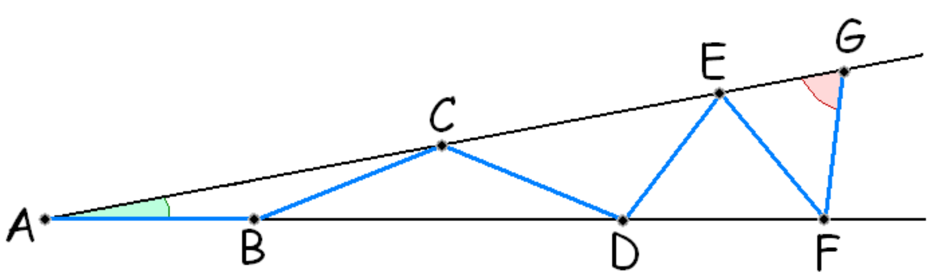
Since sides $AC = BC$ in triangle ABC, the angles opposite are also equal. This fact will give us an equation for γ.

$$\begin{aligned} 2\gamma &= (180° - 4\gamma) + \gamma \\ 2\gamma &= 180° - 3\gamma \\ 5\gamma &= 180° \\ \gamma &= 36° \end{aligned}$$

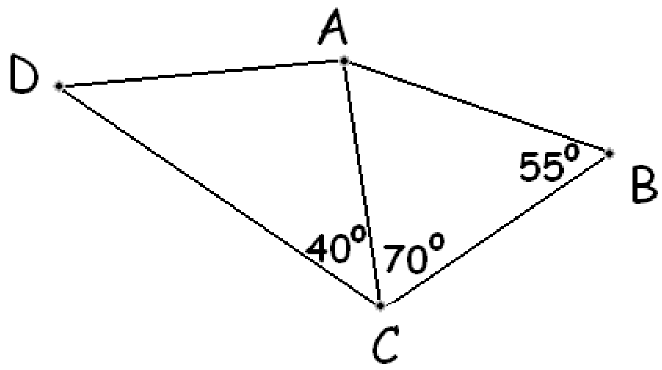
The angle at C is $36°$. the other two angles are equal, therefore they both are 2γ, $72°$. Therefore, the angles in the triangle are $36°$, $36°$, and $72°$.

Practice Problems

1. Suppose that line segments AB, BC, CD, DE, EF, FG are all of the same lengths. If the angle at point A is $10°$, what is the measure of the angle at point G?



2. In the picture given, $AD = BC$. Find the measure of the angle DAC.



Answers

1. 60° 2. 100°

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