Here is how we multiply signed numbers: first multiply the absolute values. If two integers have the same sign, their product is positive. If two integers have different signs, their product is negative. If any of the factors is zero, the product is zero.

Example 1. Compute each of the following.

a) $-3 \cdot 5$ b) $-4(-5)$ c) $10(-2)$ d) $0(-3)$ e) $-1(8)$

Solution: a) The product of a negative and a positive number is negative.
$-3 \cdot 5 = -15$

b) The product of two negative numbers is positive.
$-4(-5) = 20$

c) The product of a positive and a negative number is negative.
$10(-2) = -20$

d) If any of the factors is zero, the product is zero.
$0(-3) = 0$

e) The product of a negative and a positive number is negative.
$-1(8) = -8$

Notice that if we multiply any integer by -1, the result is the opposite of that integer. This will be very useful later.

Why do these rules work this way? Here is one possible explanation. Multiplication is defined as repeated addition. For example, $4 \cdot 7$ means that we add 7 to itself, 4 times.

$$4 \cdot 7 = 7 + 7 + 7 + 7 = 28$$

Consider now $4 \cdot (-7)$. This means that we add -7 to itself, 4 times

$$4 \cdot (-7) = -7 + (-7) + (-7) + (-7) = -28$$

The logic becomes a bit tortured, but it also works with the first factor being negative. Consider now the product $-5 \cdot 8$. We can interpret the first negative sign as repeated subtraction. So, we are subtracting 8 repeatedly, 5 times. If we feel that we don't have anything to subtract the first 8 from, we can fix that by inserting a zero. We know that adding zero will not change any value.

$$\begin{aligned} -5 \cdot 8 &= -8 - 8 - 8 - 8 - 8 \\ &= 0 - 8 - 8 - 8 - 8 - 8 && \text{to subtract is to add the opposite} \\ &= 0 + (-8) + (-8) + (-8) + (-8) + (-8) \\ &= -40 \end{aligned}$$

The most interesting case is probably when we are multiplying two negative numbers. Consider the product $-4 \cdot (-10)$. The first negative sign is interpreted as repeated subtraction, the second one is that we are subtracting negative numbers. So we are subtracting negative 10 repeatedly, 4 times. If we need something to subtract the first negative 10 from, we will just insert a zero at the beginning.

$$\begin{aligned} -4 \cdot (-10) &= 0 - (-10) - (-10) - (-10) - (-10) && \text{to subtract is to add the opposite} \\ &= 0 + 10 + 10 + 10 + 10 = 40 \end{aligned}$$

Division of integers: We will deal with zero later. For the quotient of any two non-zero integers, the rules are very simple and similar to those of multiplication. Divide the absolute values. If the the integers have the same sign, the quotient is positive. If they have different signs, the quotient is negative.

Example 2. Compute each of the following.

a) $14 \div (-2)$ b) $-24 \div (-6)$ c) $-10 \div 5$

Solution: a) Divide the absolute values. The quotient of a positive and a negative number is negative.

$14 \div (-2) = -7$

b) Divide the absolute values. The quotient of two negative numbers is positive.

$-24 \div (-6) = 4$

c) Divide the absolute values. The quotient of a negative and a positive number is negative.

$-10 \div 5 = -2$

Division is often denoted with a horizontal bar. The same computations can also be written as

$$\frac{14}{-2} = -7 \text{ and } \frac{-24}{-6} = 4 \text{ and } \frac{-10}{5} = -2$$

Why do these rules work this way? Division is defined in terms of multiplication backward. In other words,

$$\frac{20}{4} = 5 \text{ is true because } 4 \cdot 5 = 20$$

Let us apply this idea. What is the result of $14 \div (-2)$?

$$\frac{14}{-2} = \boxed{?} \text{ would be true because } -2 \cdot \boxed{?} = 14$$

Since $-2 \cdot 7$ would result in -14, we can only choose -7 to make the multiplication backward work. $-2(-7) = 14$, therefore $\frac{14}{-2} = -7$.

$$\text{Similarly, } \frac{-24}{-6} = \boxed{?} \text{ would be true because } -6 \cdot \boxed{?} = -24$$

We need to multiply -6 by a positive number to get a negative product. Only positive 4 will work, and so $\frac{-24}{-6} = 4$.

Division by Zero: Now that we understand that division is defined in terms of multiplication backward, we can easily deal with zero. The expressions $\frac{0}{3}$ and $\frac{3}{0}$ look very similar, and yet they are very different.

$$\frac{0}{3} = \boxed{?} \text{ would be true because } 3 \cdot \boxed{?} = 0$$

In this case, we can only use zero to make the multiplication backward work. Let us investigate the other case.

$$\frac{3}{0} = \boxed{?} \text{ would be true because } 0 \cdot \boxed{?} = 3$$

Now we are in trouble. If we multiply any number by zero, the product is zero. Therefore, we can not meaningfully complete this division, and so we say that $\frac{3}{0}$ is undefined. In written notation, $\frac{3}{0} = \text{undefined}$.

We have established that we cannot divide a non-zero number by zero. What about $\frac{0}{0}$?

$$\frac{0}{0} = \boxed{?} \text{ would be true because } 0 \cdot \boxed{?} = 0$$

Now the problem is that *every* number would work, because any number times zero is zero. Mathematicians prefer one clean answer as a result of an operation. We do not like an operation that results in several numbers, let alone every number! So, one fundamental rule of mathematics is that division by zero is not allowed. Indeed, division by zero is not just an error: it is one of the worst errors.

The first commandment of mathemtics: *Thou shall not divide by zero. Ever...*

Changes in Notation

With the introduction of negative numbers, our notation will have to be modified. It is a widely accepted convention that if there are several signs (operations or negative) between two numbers, a pair of parentheses must separate them.

$-2+-6$	$-5--3$	$-3\cdot-4$	$-30\div-5$
$+-$ is not allowed	$--$ is not allowed	$\cdot-$ is not allowed	$\div-$ is not allowed

For this reason, until a few decades ago, we used to put a pair of parentheses around *every negative number*.

$(-2)+(-6)$	$(-5)-(-3)$	$(-3)\cdot(-4)$	$(-30)\div(-5)$
old style	old style	old style	old style

The development of mathematical notation is an ongoing process. The most important goal in notation is clarity. As long as clarity is not jeopardized, mathematicians are in the habit of omitting things. A few decades ago we stopped putting the parentheses around the first negative number in the line or inside a parentheses, because there was no risk that we would read the sign incorrectly as subtraction. Also, there is rarely an operation sign in front of the first number.

$-2+(-6)$	$-5-(-3)$	$-3\cdot(-4)$	$-30\div(-5)$
more modern	more modern	more modern	more modern

In the case of multiplication, we can omit one more thing. Recall that multiplication is the default operation; if we see two numbers with *nothing* between them, that indicates multiplication. For example, there is no operation sign or parentheses in $2x$ or ab and yet it is clear that the operation is multiplication. Now that most negative numbers must be placed in parentheses, we can often omit the dot indicating multiplication.

$-2+(-6)$	$-5-(-3)$	$-3(-4)$	$-30\div(-5)$
cannot omit anything	cannot omit anything	we can omit the dot	cannot omit anything

The most common modern style is minimalistic, ommitting as much as possible, as long as confusion is avoided. This can lead to apparent irregularities in notation. For example, our notation will be $2(-3)$, but when we swap the two factors, it will be $-3\cdot 2$.

Practice Problems

1. $-2+7$
2. $-7-(-4)$
3. $12\div(-2)$
4. $5(-3)$
5. $-8\cdot 0$
6. $-3-(-10)$
7. $-20\div 0$
8. $-12\div 3$
9. $-4\cdot 7$
10. $-6-|-7|$
11. $-3\div(-3)$
12. $|9|+(-1)$
13. $-3-0$
14. $0(-4)$
15. $\dfrac{-5}{0}$
16. $0\div(-1)$
17. $9+|-1|$
18. $|9+(-1)|$

Answers

Practice Problems

1. 5 2. -3 3. -6 4. -15 5. 0 6. 7 7. undefined 8. -4 9. -28 10. -13 11. 1 12. 8

13. -3 14. 0 15. undefined 16. 0 17. 10 18. 8

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