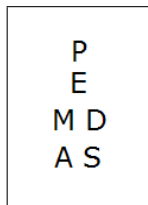


The order of operations rule is an agreement among mathematicians, it simplifies notation.



P stands for parentheses, E for exponents, M and D for multiplication and division, A and S for addition and subtraction. Notice that M and D are written next to each other. This is to suggest that multiplication and division are equally strong. Similarly, A and S are positioned to suggest that addition and subtraction are equally strong. This is the hierarchy, and there are two basic rules.

1. **Between two operations that are on different levels of the hierarchy, we start with the operation that is higher.** For example, between a division and a subtraction, we start with the division since it is higher in the hierarchy than subtraction.
2. **Between two operations that are on the same level of the hierarchy, we start with the operation that comes first from reading left to right.**

These basic rules pretty much cover all possible situation. At every step, we execute only one operation, and replace the result by the expression indicating the operation. Let us see a few examples.

Example 1 $20 - 3 \cdot 4$

Solution: We observe two operations, a subtraction and a multiplication. Multiplication is higher in the hierarchy than subtraction, so we start there.

$$\begin{aligned} 20 - 3 \cdot 4 &= && \text{multiplication} \\ 20 - 12 &= && \text{subtraction} \\ &= && \boxed{8} \end{aligned}$$

Example 2 $36 \div 3 \cdot 2$

Solution: it is a very common mistake to start with the multiplication. The letters M and D are in the same line because they are equally strong; among them we proceed left to right. From left to right, the division comes first

$$\begin{aligned} 36 \div 3 \cdot 2 &= && \text{division} \\ 12 \cdot 2 &= && \text{multiplication} \\ &= && \boxed{24} \end{aligned}$$

Example 3 $36 \div 2 \div 2$

Solution: It is essential to perform these two divisions left to right. If we proceeded differently, we would get a different result.

$$\begin{aligned} 36 \div 2 \div 2 &= && \text{first division from left} \\ 18 \div 2 &= && \text{division} \\ &= && \boxed{9} \end{aligned}$$



Sample Problems

Simplify each of the following expressions by applying the order of operations agreement.

1. $2 \cdot 3^2 - (6^2 - 2 \cdot 5) \div 2$

4. $8^2 - 3^2$

6. $(3^3 - 4 \cdot 5 + 2)^2$

2. $18 - 7 - 3$

5. $(8 - 3)^2$

7. $\frac{3 + 2(20 - 3^2 - 5)}{3^2 - 2^2} + 4^1$

3. $5^2 - 2(10 - 2^2)$



Practice Problems

Simplify each of the following expressions by applying the order of operations agreement.

1. $2 \cdot 5^2 - (6 \cdot 5 - 3^2) \div 3$

7. $\frac{(5 - 3)^2}{2^2}$

12. $30 - (2(15 - 2^3) - 2^2)$

2. $10^2 - 7^2$

8. $120 \div 6 \cdot 2$

13. $4(3(2(2^2 - 1) + 1) - 1) + 5$

3. $(10 - 7)^2$

9. $\left((7 - 4)^2 - 5\right)^2 - 1$

14. $\frac{2(3^3 - 4 \cdot 5) - 2^2}{4^2 - (3^2 + 2)}$

4. $20 - 7 - 1$

10. $\frac{22 - 3^2 + 2(20 - 3^2 - 5)}{3^2 - 2^2}$

5. $2^3 - (11 - 3^2)^2$

11. $\frac{5 + (5^2 - 3^2)}{3^2 - 2 \cdot 1^8}$

15. $\left(\left(\left(\left((7 - 4)^2 - 7\right)^2 - 3\right)^2 + 5\right)^2\right)^2$

6. $\frac{5^2 - 3^2}{2^2}$



Enrichment

1. Place one or more pairs of parentheses into the expression on the left-hand side to make the equation true.

$$12 - 2 \cdot 3 - 1^2 + 2 - 3 + 4 = 20$$



Answers

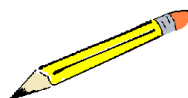
Sample Problems

1. 5 2. 8 3. 13 4. 55 5. 25 6. 81 7. 7

Practice Problems

1. 43 2. 51 3. 9 4. 12 5. 4 6. 4 7. 1 8. 40 9. 15 10. 5 11. 3
12. 20 13. 85 14. 2 15. 36

Sample Problems



Solutions

Simplify each of the following expressions by applying the order of operations agreement.

$$1. 2 \cdot 3^2 - (6^2 - 2 \cdot 5) \div 2$$

Solution: We start with the parentheses. We will work within the parentheses until the entire expression within it becomes one number. In the parentheses, there is an exponentiation, a subtraction, and a multiplication. Since it is stronger, we start with the exponent.

$$\begin{aligned} 2 \cdot 3^2 - (6^2 - 2 \cdot 5) \div 2 &= \text{exponent within parentheses} \\ 2 \cdot 3^2 - (36 - 2 \cdot 5) \div 2 &= \text{multiplication within parentheses} \\ 2 \cdot 3^2 - (36 - 10) \div 2 &= \text{subtraction within parentheses} \\ 2 \cdot 3^2 - (26) \div 2 &= \text{we may drop parentheses now} \\ 2 \cdot 3^2 - 26 \div 2 &= \end{aligned}$$

Now that there is no parentheses, we perform all exponents, left to right. There is only one, so we have

$$\begin{aligned} 2 \cdot 3^2 - 26 \div 2 &= \text{exponent} \\ 2 \cdot 9 - 26 \div 2 &= \end{aligned}$$

Now we execute all multiplications, divisions, left to right

$$\begin{aligned} 2 \cdot 9 - 26 \div 2 &= \text{multiplication} \\ 18 - 26 \div 2 &= \text{division} \\ 18 - 13 &= \text{subtraction} \\ &= \boxed{5} \end{aligned}$$

$$2. 18 - 7 - 3$$

Solution: It is a common mistake to subtract 4 from 18. This is not what order of operations tell us to do. The two subtractions have to be performed left to right.

$$\begin{aligned} 18 - 7 - 3 &= \text{first subtraction from left} \\ 11 - 3 &= \text{subtraction} \\ &= \boxed{8} \end{aligned}$$

$$3. 5^2 - 2(10 - 2^2)$$

Solution: We start with the parentheses

$$\begin{aligned} 5^2 - 2(10 - 2^2) &= \text{exponent in parentheses} \\ 5^2 - 2(10 - 4) &= \text{subtraction in parentheses} \\ 5^2 - 2(6) &= \text{drop parentheses} \\ 5^2 - 2 \cdot 6 &= \text{exponents} \\ 25 - 2 \cdot 6 &= \text{multiplication} \\ 25 - 12 &= \text{subtraction} \\ &= \boxed{13} \end{aligned}$$

4. $8^2 - 3^2$

Solution: There are three operations, two exponents and a subtraction. We start with the exponents, left to right.

$$\begin{aligned} 8^2 - 3^2 &= && \text{first exponent from left} \\ 64 - 3^2 &= && \text{exponent} \\ 64 - 9 &= && \text{subtraction} \\ &= && \boxed{55} \end{aligned}$$

5. $(8 - 3)^2$

Solution: We start with the parentheses

$$\begin{aligned} (8 - 3)^2 &= && \text{subtraction in parentheses} \\ (5)^2 &= && \text{drop parentheses} \\ 5^2 &= && \text{exponents} \\ &= && \boxed{25} \end{aligned}$$

This problem and the previous one tells us a very important thing: $a^2 - b^2$ and $(a - b)^2$ are different expressions! In $a^2 - b^2$ we first square a and b and then subtract. In $(a - b)^2$ we first subtract b from a and then square the difference.

6. $(3^3 - 4 \cdot 5 + 2)^2$

Solution: We will work within the parentheses until it becomes a number. Within the parentheses, we start with the exponents.

$$\begin{aligned} (3^3 - 4 \cdot 5 + 2)^2 &= && \text{exponents within parentheses} \\ (27 - 4 \cdot 5 + 2)^2 &= && \text{multiplication within parentheses} \\ (27 - 20 + 2)^2 &= && \end{aligned}$$

There is an addition and a subtraction in the parentheses. **It is not true that addition comes before subtraction!** Addition and subtraction are equally strong; we execute them left to right.

$$\begin{aligned} (27 - 20 + 2)^2 &= && \text{subtraction within parentheses} \\ (7 + 2)^2 &= && \text{addition within parentheses} \\ (9)^2 &= && \text{drop parentheses} \\ 9^2 &= && \text{exponents} \\ &= && \boxed{81} \end{aligned}$$

$$7. \frac{3 + 2(20 - 3^2 - 5)}{3^2 - 2^2} + 4^1$$

Solution: The division bar stretching over entire expressions is a case of the **invisible parentheses**. It instructs us to work out the top until we obtain a number, the bottom until we obtain a number, and finally divide. The invisible parentheses here means

$$\frac{3 + 2(20 - 3^2 - 5)}{3^2 - 2^2} = [3 + 2(20 - 3^2 - 5)] \div [3^2 - 2^2]$$

And now we see that the invisible parentheses was developed to simplify notation. We will start with the top. Naturally, we stay within the parentheses until they disappear.

$$\begin{aligned} \frac{3 + 2(20 - 3^2 - 5)}{3^2 - 2^2} + 4^1 &= \text{exponent in parentheses} \\ \frac{3 + 2(20 - 9 - 5)}{3^2 - 2^2} + 4^1 &= \text{first subtraction from left in parentheses} \\ \frac{3 + 2(11 - 5)}{3^2 - 2^2} + 4^1 &= \text{subtraction in parentheses} \\ \frac{3 + 2(6)}{3^2 - 2^2} + 4^1 &= \text{drop parentheses} \\ \frac{3 + 2 \cdot 6}{3^2 - 2^2} + 4^1 &= \text{multiplication on top} \\ \frac{3 + 12}{3^2 - 2^2} + 4^1 &= \text{addition on top} \\ \frac{15}{3^2 - 2^2} + 4^1 &= \end{aligned}$$

Now we work out the bottom, applying order of operations

$$\begin{aligned} \frac{15}{3^2 - 2^2} + 4^1 &= \text{first exponent from left to right} \\ \frac{15}{9 - 2^2} + 4^1 &= \text{exponent} \\ \frac{15}{9 - 4} + 4^1 &= \text{subtraction} \\ \frac{15}{5} + 4^1 &= \text{same as} \\ 15 \div 5 + 4^1 &= \end{aligned}$$

We now have a division, an addition, and an exponent. We start with the exponent.

$$\begin{aligned} 15 \div 5 + 4^1 &= \text{exponent, } 4^1 = 4 \\ 15 \div 5 + 4 &= \text{division} \\ 3 + 4 &= \text{addition} \\ &= \boxed{7} \end{aligned}$$

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