The order of operations rule is an agreement among mathematicians, it simplifies notation.

P
E
M D
A S

P stands for parentheses, E for exponents, M and D for multiplication and division, A and S for addition and subtraction. Notice that M and D are written next to each other. This is to suggest that multiplication and division are equally strong. Similarly, A and S are positioned to suggest that addition and subtraction are equally strong. This is the hierarchy, and there are two basic rules.

1. **Between two operations that are on different levels of the hierarchy, we start with the operation that is higher.** For example, between a division and a subtraction, we start with the division since it is higher in the hierarchy than subtraction.

2. **Between two operations that are on the same level of the hierarchy, we start with the operation that comes first from reading left to right.**

These basic rules pretty much cover all possible situation. At every step, we execute only one operation, and replace the result by the expression indicating the operation. Let us see a few examples.

Example 1 $20 - 3 \cdot 4$

Solution: We observe two operations, a subtraction and a multiplication. Multiplication is higher in the hierarchy than subtraction, so we start there.

$$\begin{aligned} 20 - 3 \cdot 4 &= & &\text{multiplication} \\ 20 - 12 &= & &\text{subtraction} \\ &= \boxed{8} \end{aligned}$$

Example 2 $36 \div 3 \cdot 2$

Solution: it is a very common mistake to start with the multiplication. The letters M and D are in the same line because they are equally strong; among them we proceed left to right. From left to right, the division comes first

$$\begin{aligned} 36 \div 3 \cdot 2 &= & &\text{division} \\ 12 \cdot 2 &= & &\text{multiplication} \\ &= \boxed{24} \end{aligned}$$

Example 3 $36 \div 2 \div 2$

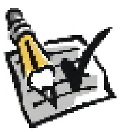
Solution: It is essential to perform these two divisions left to right. If we proceeded differently, we would get a different result.

$$\begin{aligned} 36 \div 2 \div 2 &= & &\text{first division from left} \\ 18 \div 2 &= & &\text{division} \\ &= \boxed{9} \end{aligned}$$

Sample Problems

Simplify each of the following expressions by applying the order of operations agreement.

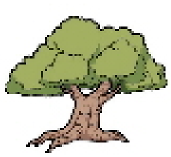
1. $2 \cdot 3^2 - \left(6^2 - 2 \cdot 5\right) \div 2$
2. $18 - 7 - 3$
3. $5^2 - 2\left(10 - 2^2\right)$
4. $8^2 - 3^2$
5. $(8-3)^2$
6. $\left(3^3 - 4 \cdot 5 + 2\right)^2$
7. $\dfrac{3 + 2\left(20 - 3^2 - 5\right)}{3^2 - 2^2} + 4^1$



Practice Problems

Simplify each of the following expressions by applying the order of operations agreement.

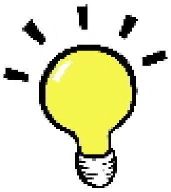
1. $2 \cdot 5^2 - \left(6 \cdot 5 - 3^2\right) \div 3$
2. $10^2 - 7^2$
3. $(10-7)^2$
4. $20 - 7 - 1$
5. $2^3 - \left(11 - 3^2\right)^2$
6. $\dfrac{5^2 - 3^2}{2^2}$
7. $\dfrac{(5-3)^2}{2^2}$
8. $120 \div 6 \cdot 2$
9. $\left((7-4)^2 - 5\right)^2 - 1$
10. $\dfrac{22 - 3^2 + 2\left(20 - 3^2 - 5\right)}{3^2 - 2^2}$
11. $\dfrac{5 + \left(5^2 - 3^2\right)}{3^2 - 2 \cdot 1^8}$
12. $30 - \left(2\left(15 - 2^3\right) - 2^2\right)$
13. $4\left(3\left(2(2^2 - 1) + 1\right) - 1\right) + 5$
14. $\dfrac{2\left(3^3 - 4 \cdot 5\right) - 2^2}{4^2 - (3^2 + 2)}$



Enrichment

1. Place one or more pairs of parentheses into the expression on the left-hand side to make the equation true.

$$12 - 2 \cdot 3 - 1^2 + 2 - 3 + 4 = 20$$



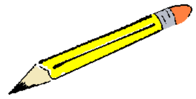
Answers

Sample Problems

1. 5 2. 8 3. 13 4. 55 5. 25 6. 81 7. 7

Practice Problems

1. 43 2. 51 3. 9 4. 12 5. 4 6. 4 7. 1 8. 40 9. 15 10. 5 11. 3
12. 20 13. 85 14. 2

Sample Problems  Solutions

Simplify each of the following expressions by applying the order of operations agreement.

1. $2 \cdot 3^2 - \left(6^2 - 2 \cdot 5\right) \div 2$

   Solution: We start with the parentheses. We will work within the parentheses until the entire expression within it becomes one number. In the parentheses, there is an exponentiation, a subtraction, and a multiplication. Since it is stronger, we start with the exponent.

$$\begin{array}{rcl} 2 \cdot 3^2 - \left(6^2 - 2 \cdot 5\right) \div 2 & = & \text{exponent within parentheses} \\ 2 \cdot 3^2 - (36 - 2 \cdot 5) \div 2 & = & \text{multiplication within parentheses} \\ 2 \cdot 3^2 - (36 - 10) \div 2 & = & \text{subtraction within parentheses} \\ 2 \cdot 3^2 - (26) \div 2 & = & \text{we may drop parentheses now} \\ 2 \cdot 3^2 - 26 \div 2 & = & \end{array}$$

   Now that there is no parentheses, we perform all exponents, left to right. There is only one, so we have

$$\begin{array}{rcl} 2 \cdot 3^2 - 26 \div 2 & = & \text{exponent} \\ 2 \cdot 9 - 26 \div 2 & = & \end{array}$$

   Now we execute all multiplications, divisions, left to right

$$\begin{array}{rcl} 2 \cdot 9 - 26 \div 2 & = & \text{multiplication} \\ 18 - 26 \div 2 & = & \text{division} \\ 18 - 13 & = & \text{subtraction} \\ & = & \boxed{5} \end{array}$$

2. $18 - 7 - 3$

   Solution: It is a common mistake to subtract 4 from 18. This is not what order of operations tell us to do. The two subtractions have to be performed left to right.

$$\begin{array}{rcl} 18 - 7 - 3 & = & \text{first subtraction from left} \\ 11 - 3 & = & \text{subtraction} \\ & = & \boxed{8} \end{array}$$

3. $5^2 - 2\left(10 - 2^2\right)$

   Solution: We start with the parentheses

$$\begin{array}{rcl} 5^2 - 2\left(10 - 2^2\right) & = & \text{exponent in parentheses} \\ 5^2 - 2\,(10 - 4) & = & \text{subtraction in parentheses} \\ 5^2 - 2\,(6) & = & \text{drop parentheses} \\ 5^2 - 2 \cdot 6 & = & \text{exponents} \\ 25 - 2 \cdot 6 & = & \text{multiplication} \\ 25 - 12 & = & \text{subtraction} \\ & = & \boxed{13} \end{array}$$

4. $8^2 - 3^2$

Solution: There are three operations, two exponents and a subtraction. We start with the exponents, left to right.

$$\begin{aligned} 8^2 - 3^2 &= && \text{first exponent from left} \\ 64 - 3^2 &= && \text{exponent} \\ 64 - 9 &= && \text{subtraction} \\ &= \boxed{55} \end{aligned}$$

5. $(8-3)^2$

Solution: We start with the parentheses

$$\begin{aligned} (8-3)^2 &= && \text{subtraction in parentheses} \\ (5)^2 &= && \text{drop parentheses} \\ 5^2 &= && \text{exponents} \\ &= \boxed{25} \end{aligned}$$

This problem and the previous one tells us a very important thing: $a^2 - b^2$ and $(a-b)^2$ are different expressions! In $a^2 - b^2$ we first square a and b and the subtract. In $(a-b)^2$ we first subtract b from a and then square the difference.

6. $\left(3^3 - 4 \cdot 5 + 2\right)^2$

Solution: We will work within the parentheses until it becomes a number. Within the parentheses, we start with the exponents.

$$\begin{aligned} \left(3^3 - 4 \cdot 5 + 2\right)^2 &= && \text{exponents within parentheses} \\ (27 - 4 \cdot 5 + 2)^2 &= && \text{multiplication within parentheses} \\ (27 - 20 + 2)^2 &= \end{aligned}$$

There is an addition and a subtraction in the parentheses. **It is not true that addition comes before subtraction!** Addition and subtraction are equally strong; we execute them left to right.

$$\begin{aligned} (27 - 20 + 2)^2 &= && \text{subtraction within parentheses} \\ (7+2)^2 &= && \text{addition within parentheses} \\ (9)^2 &= && \text{drop parentheses} \\ 9^2 &= && \text{exponents} \\ &= \boxed{81} \end{aligned}$$

7. $\dfrac{3+2\left(20-3^2-5\right)}{3^2-2^2}+4^1$

Solution: The division bar stretching over entire expressions is a case of the **invisible parentheses**. It instruct us to work out the top until we obtain a number, the bottom until we obtain a number, and finally divide. The invisible parentheses here means

$$\frac{3+2\left(20-3^2-5\right)}{3^2-2^2}=\left[3+2\left(20-3^2-5\right)\right]\div\left[3^2-2^2\right]$$

And now we see that the invisible parentheses was developed to simplify notation. We will start with the top. Naturally, we stay within the parentheses until they disappear.

$$\begin{aligned}
\frac{3+2\left(20-3^2-5\right)}{3^2-2^2}+4^1 &= && \text{exponent in parentheses}\\
\frac{3+2\left(20-9-5\right)}{3^2-2^2}+4^1 &= && \text{first subtraction from left in parentheses}\\
\frac{3+2\left(11-5\right)}{3^2-2^2}+4^1 &= && \text{subtraction in parentheses}\\
\frac{3+2\left(6\right)}{3^2-2^2}+4^1 &= && \text{drop parentheses}\\
\frac{3+2\cdot 6}{3^2-2^2}+4^1 &= && \text{multiplication on top}\\
\frac{3+12}{3^2-2^2}+4^1 &= && \text{addition on top}\\
\frac{15}{3^2-2^2}+4^1 &=
\end{aligned}$$

Now we work out the bottom, applying order of operations

$$\begin{aligned}
\frac{15}{3^2-2^2}+4^1 &= && \text{first exponent from left to right}\\
\frac{15}{9-2^2}+4^1 &= && \text{exponent}\\
\frac{15}{9-4}+4^1 &= && \text{subtraction}\\
\frac{15}{5}+4^1 &= && \text{same as}\\
15\div 5+4^1 &=
\end{aligned}$$

We now have a division, an addition, and an exponent. We start with the exponent.

$$\begin{aligned}
15\div 5+4^1 &= && \text{exponent, } 4^1=4\\
15\div 5+4 &= && \text{division}\\
3+4 &= && \text{addition}\\
&= && \boxed{7}
\end{aligned}$$

