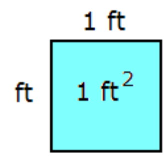
The **area** of a geometric object is a measurement of its surface.

While we could think about perimeter as a fencing problem, area can be thought of as follows. Suppose a geometric object is a room. How much rug do we need to buy to cover the entire room? Understanding and remembering the area formulas are probably easier if we know how they were derived.

Definition: The area of a 1 feet by 1 feet square is defined to be $1\,\mathrm{ft}^2$ (square-feet). (Similar definitions can be formulated with mi^2, cm^2, in^2, etc.) The area of an object, measured in ft^2, is the number of $1\,\mathrm{ft}$ by $1\,\mathrm{ft}$ square needed to cover the object, cutting and pasting allowed.

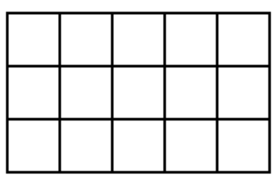


Area is not a length like perimeter. Area is always measured in ft^2, mi^2, cm^2, in^2, etc., and is usually denoted by A.

Part 1 - Rectangles

Theorem: The area of a rectangle with sides x and y is $A = xy$.

Proof: Consider rectangle with sides $3\,\mathrm{m}$ and $5\,\mathrm{m}$. The area of this rectangle will be as many m^2 as many $1\,\mathrm{m}$ by $1\,\mathrm{m}$ m squares are needed to cover it. Once we place this grid on the rectangle, it is easy to see, just how many squares we need.



We used exactly 15 squares to cover the rectangle, and so the area is $15\,\mathrm{m}^2$.

Mathematicians also proved that the formula is true even if the sides of the rectangle are not integers.
It is interesting to see that we basically counted how many square meters we have. A computation for the area that includes the units is slightly different. Instead of counting square meters, we literally multiply meter by meter.

$$A = ab = 3\,\mathrm{m}\,(5\,\mathrm{m}) = 15\,\mathrm{m}^2$$

Area computation will always yield units such as m^2 (square meters), or in^2 (square inches), or mi^2 (square miles) and so on.

Example 1: Find the area of a rectangle with sides 13 in and 7 in.

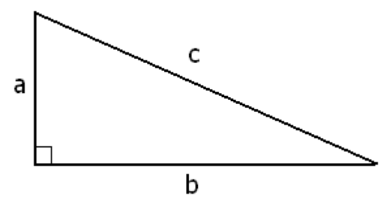
Solution: We apply the formula $A = xy$.

$$A = xy = 13\,\mathrm{in}\,(7\,\mathrm{in}) = 91\,\mathrm{in}^2$$

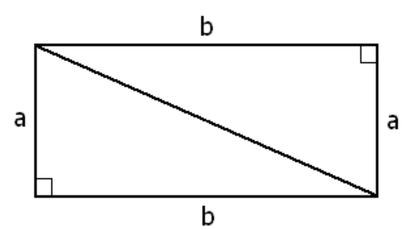
Part 2 - Triangles

The following few area formulas will demonstrate how mathematicians work: we will use already proven results to come up with new formulas. We will first consider right triangles.

Theorem: The area of a right triangle with sides a, b, and c (where c is the longest side) is $A = \dfrac{ab}{2}$.



Proof: It is very easy to see that every right triangle is basically half of a rectangle. We can make a rectangle if we use two identical right triangles as shown on the picture below.



Since the rectangle's sides are a and b, its area is ab. The area of our triangle must be half of it. Thus $A = \dfrac{ab}{2}$.
Notice that we never used the length of the longest side, c.

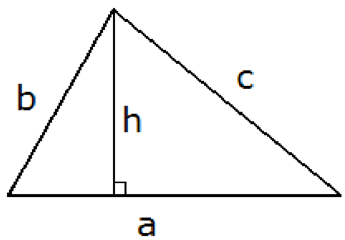
Example 2: Find the area of the right triangle with sides 5 m, 12 m, and 13 m long.

Solution: It is important to know that the largest side, 13 m long, is not needed for this computation. With labeling $a = 5\,\mathrm{m}$ and $b = 12\,\mathrm{m}$, the area is

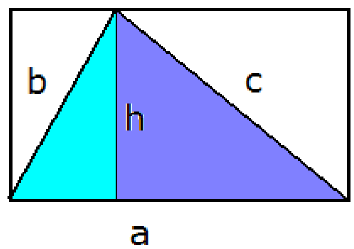
$$A = \frac{ab}{2} = \frac{5\,\mathrm{m}\,(12\,\mathrm{m})}{2} = \frac{60\,\mathrm{m}^2}{2} = 30\,\mathrm{m}^2$$

Let us now consider general triangles.

Theorem: The area of a general triangle with sides a, b, c and height h as shown on the picture below is $A = \dfrac{ah}{2}$.

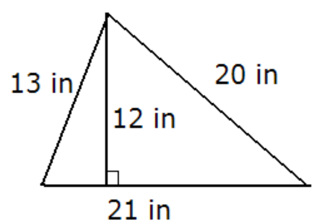


Proof: As before, we will use a previously obtained result. Since the general triangle no longer has a right angle, we create it by drawing in the altitude or height belonging to the side a. Now we split our triangle into two right triangles, and each of them is half of a rectangle.



Our triangle makes up for half of a rectangle, with sides a and h. Thus $A = \dfrac{ah}{2}$.

Example 3: Find the area of the triangle shown on the picture below.

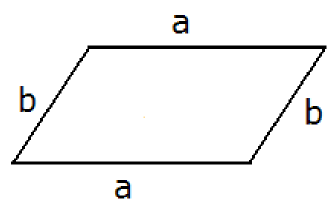


Solution: It is important to notice that we will not need all the information given. We apply the area-formula.

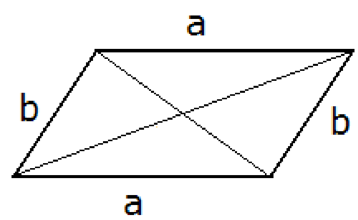
$$A = \frac{ah}{2} = \frac{21\,\text{in}\,(12\,\text{in})}{2} = \frac{252\,\text{in}^2}{2} = 126\,\text{in}^2$$

Part 3 - Parallelograms

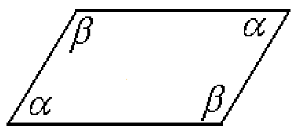
Definition: A parallelogram is a four sided polygon with two pairs of parallel sides.



It is a proven fact that the opposite sides of a parallelogram are of equal length. This is not part of the definition, but it is an important property that we need to remember. Also, we could prove that the diagonals of a parallelogram always bisect each other.



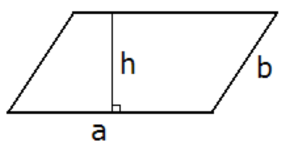
Another important property of parallelograms is the connection between its angles. In every parallelogram, the opposite angles are equal, and the two angles along each side add up to 180°. We call two such angles supplemental.



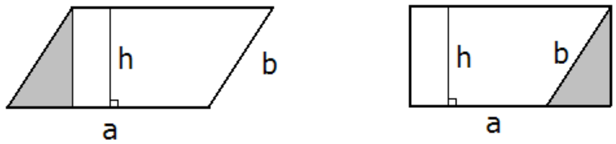
$$\alpha + \beta = 180^\circ$$

This is the property that enables us to easily compute the area of the parallelogram.

Theorem: The area of a parallelogram with sides a, b and height h belonging to a is $A = ah$.

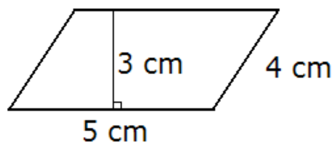


Proof: We will use (surprise, surprise!) a previously proven result. If we cut off a triangle and paste it back as show on the picture below, we obtain a rectangle.



Thus the area of the parallelogram equals to the area of a rectangle with sides a and h.

Example 4: Find the area of the parallelogram shown on the picture below.

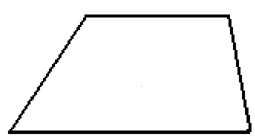


Solution: We apply the formula for the area of a parallelogram.

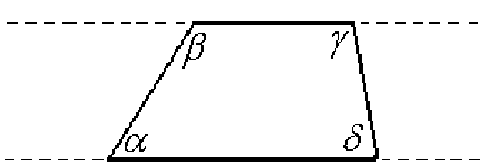
$$A = ah = 5\,\text{cm}\,(3\,\text{cm}) = 15\,\text{cm}^2$$

Part 4 - Trapezoids

Definition: A **trapezoid** is a four sided polygon with one pair of parallel sides.



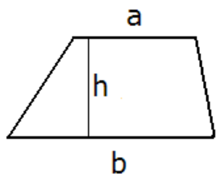
In proving the area formula for trapezoids, we will use a property about its angles. It is proven that the two angles along a side connetcting two parallel sides add up to 180°.



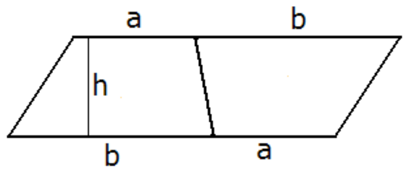
$$\alpha + \beta = 180^{\circ}$$

$$\gamma + \delta = 180^{\circ}$$

Theorem: The area of a trapezoid, with sides and height labeled as on the picture below, is $A = \dfrac{a+b}{2}h$.

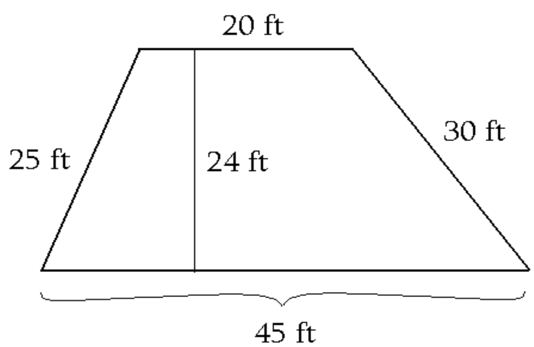


Proof: If we use two identical trapezoids, we can make a parallelogram as shown on the picture below.



We already know that the area of this parallelogram is $A = (a+b)\,h$. Since our trapezoid is exactly half of the parallelogram, its area is $A = \dfrac{(a+b)\,h}{2}$.

Example 5: Find the area of the trapezoid shown on the picture below.

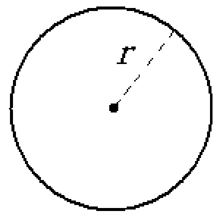


Solution: We apply the formula. It is important to notice that we will not need all data given. For the area, we only need the lengths on the parallel sides and the height connecting them. With that data, we use the formula $A = \dfrac{a+b}{2}h$.

$$A = \frac{a+b}{2}h = \frac{20\,\text{ft} + 45\,\text{ft}}{2}(24\,\text{ft}) = \frac{65\,\text{ft}}{2}(24\,\text{ft}) = 780\,\text{ft}^2$$

Part 5 - Circles

Definition: A circle is the set of all points in a plane that are equidistant to a fixed point. That equal distance is called the radius of the circle, that fixed point is called the center of the circle.



Theorem: The area of a circle with radius r is $A = \pi r^2$.

At this level of mathematics, we do not have the tools necessary to prove this. The proof actually uses (again) the formula for the area of a rectangle. Instead of a circle, we use just half of it, and then we multiply the result by 2. The basic idea is to first find an estimation for the area, using rectangles

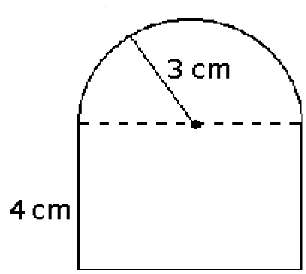
We CAN compute the grey area since it is composed of rectangles. But this is a crude underestimation of the actual area. The trick is that the more rectangles we use, the more accurate the approximation becomes. Using more and more rectangles to estimate the area of the semi-circle, the approximation becomes better and better.

Calculus offers tools to find a unique number these approximations approach. This number is the area.

Example 6: Find the area of a circle of radius 7 ft.
Solution: We apply the formula $A = \pi r^2$.

$$A = \pi r^2 = \pi\,(7\,\text{ft})^2 \approx 153.94\,\text{ft}^2$$

Example 7: Find the area of the figure shown on the picture below.



Solution: The area is the sum of the areas of a rectangle and a semi-circle. We apply the appropriate formulas and then add the results. The horizontal side of the rectangle is 6 cm since we can fit exactly two radii on it.

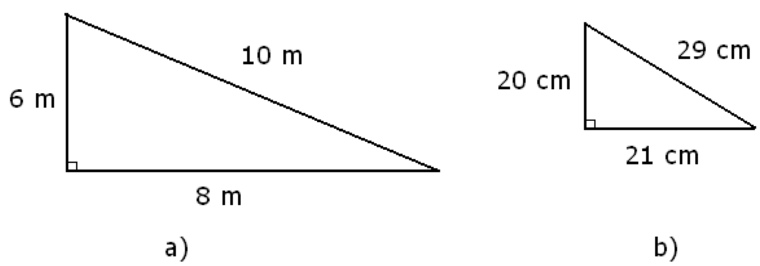
$$\begin{aligned} A_{\text{rectangle}} &= ab = 4\,\text{cm}\,(6\,\text{cm}) = 24\,\text{cm}^2 \\ A_{\text{semicircle}} &= \frac{\pi r^2}{2} = \frac{\pi\,(3\,\text{cm})^2}{2} \approx 14.137\,\text{cm}^2 \\ A &= A_{\text{rectangle}} + A_{\text{semicircle}} \approx 24\,\text{cm}^2 + 14.137\,\text{cm}^2 = 38.137\,\text{cm}^2 \end{aligned}$$

Practice Problems

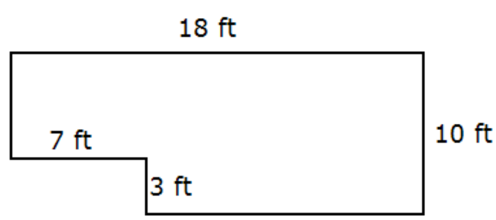
1. The sides of a rectangle are given below. Find the area of the rectangle. Include units in your answer.

   a) 4 in and 11 in b) 15 cm and 8 cm

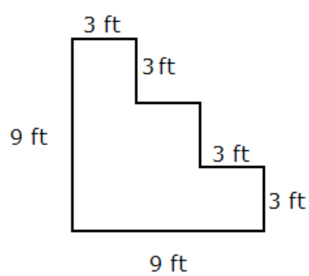
2. Find the area of the right triangles shown on the picture below.



3. Find the area of the figure shown on the picture below. Include units in your answer.



4. Find the area of the figure shown on the picture below. Include units in your answer.



5. Find the area of a circle with radius 7 in long.

Practice Problems - Answers

1. a) $A = 44\,\text{in}^2$ b) $A = 60\,\text{in}^2$
2. a) $A = 24\,\text{m}^2$ b) $A = 210\,\text{cm}^2$
3. $A = 159\,\text{ft}^2$
4. $A = 54\,\text{ft}^2$
5. $A = 49\pi\,\text{in}^2 \approx 153.938\ \text{in}^2$