Definition: Suppose that N and m are any two integers. If there exists an integer k such that $N = mk$, then we say that m is a **factor** or **divisor** of N. We also say that N is a **multiple** of m or that N is **divisible** by m. Notation: $m|N$

For example, 3 is a factor of 15 because there exists another integer (namely 5) so that $3 \cdot 5 = 15$. Notation: $3|15$.

Example 1 Label each of the following statements as true or false.

a) 2 is a factor of 10 b) 3 is divisible by 3 c) 14 is a factor of 7 d) 0 is a multiple of 5

e) every integer is divisible by 1 f) every integer n is divisible by n

Solution: a) $10 = 2 \cdot 5$ and so 2 is a factor of 10. This statement is true.

b) $3 = 3 \cdot 1$ and so 3 is divisible by 3. This statement is true.

c) $14 = 7 \cdot 2$ and so 14 is a multiple of 7, not a factor. Can we find an integer k so that $7 = 14 \cdot k$? This is not possible. $k = \dfrac{1}{2}$ would work, but $\dfrac{1}{2}$ is not an integer. This statement is false.

d) Since $0 = 5 \cdot 0$, it is indeed true that 0 is a multiple of 5. This statement is true.

e) For any integer n, $n = n \cdot 1$ and so every integer n is divisible by 1. This statement is true.

f) For any integer n, $n = 1 \cdot n$ and so every integer n is divisible by n. This statement is true.

Example 2 List all positive factors of the number 28.

Solution: We start counting, starting at 1.
Is 1 a divisor of 28? Yes, because $28 = 1 \cdot 28$.
We note both factors we found.

28	
1	28

We continue counting. Is 2 a divisor of 28?
Yes, because $28 = 2 \cdot 14$.
We note both factors we found.

28	
1	28
2	14

We continue counting. Is 3 a divisor of 28? No.
We can divide 28 by 3 and the answer is not an integer.

We continue counting. Is 4 a divisor of 28? Yes, because $28 = 4 \cdot 7$. We note both factors we found.

28	
1	28
2	14
4	7

We continue counting. Is 5 a divisor of 28? No. (We can check with the calculator.) Is 6 a divisor of 28? No. Now we arrive to 7, a number that is already listed as a factor. That's our signal that we have found all of the divisors of 28. We list the divisors in order:

factors of 28: $\boxed{1, 2, 4, 7, 14, 28}$

Discussion: What do you think about the argument shown below?

3 is a divisor of 21 because there exists another integer, namely 7 so that $21 = 3 \cdot 7$. As we established that 3 is a divisor of 21, we also found that 7 is also a divisor of 21. In other words, divisors always come in pairs. For example, 28 has six divisors that we found in three pairs: 1 with 28, 2 with 14, and 4 with 7. Consequently, every positive integer has an even number of positive divisors.

Definition: An integer is a **prime number** if it has exactly two divisors: 1 and itself.

For example, 37 is a prime number.

We will later prove all of the following statements. They will cut down on the work as we look for divisors of a number.

Theorem: A number n is divisible by 2 if its last digit is 0, 2, 4, 6, or 8.
A number n is divisible by 5 if its last digit is 0, or 5.
A number n is divisible by 4 if the two-digit number formed of its last two digits is divisible by 4.
A number n is divisible by 3 if the sum of its all digits is divisible by 3.
A number n is divisible by 9 if the sum of its all digits is divisible by 9.

Practice Problems

1. List all the factors of 48.
2. Which of the following is NOT a prime number? $53, 73, 91, 101, 139$
3. Consider the following numbers. $128,\ 80, 75, 270,\ 64$
   a) Find all numbers on the list that are divisible by 5.
   b) Find all numbers on the list that are divisible by 3.
   c) Find all numbers on the list that are divisible by 4.

Answers

1. $1, 2, 3, 4, 6, 8, 12, 16, 24, 48$ 2. 91 3. a) $80, 75, 270$ b) $75, 270$ c) $128, 80, 64$