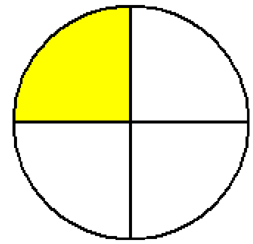
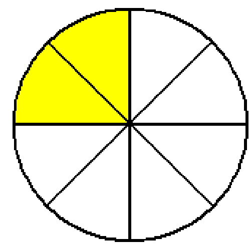
So far, we didn't look at a standalone fraction on its own. We only defined a fraction as expressing part of something. We will continue to do this. A warning to the reader: do not read this section while hungry. We will be talking a lot about cakes and pizzas.



Suppose that Ann, Bethany, Cecile and Desire are about to share a pizza. They cut the pizza into four equal slices. Ann happily looks at her slice (the shaded region). What fraction can express this slice? The answer is $\frac{1}{4}$. We are talking about one slice, so big so that four equal slices make up the whole.



Just before they start to eat, four other friends arrive. They decide to share, so everyone cuts their own slice in two equal pieces. Before she gives her friend half of her food, Ann looks at her plate. Her share now looks like this. What fraction could express this? Since we cut each slice in two, the four slices became eight slices. So, we are now looking at $\frac{2}{8}$.

The fractions $\frac{1}{4}$ and $\frac{2}{8}$ express the same part of a given quantity. Such fractions are called **equivalent** to each other. Given any fraction, we can create infitely many equaivalent fractions by multiplying both numerator and denominator by the same non-zero number. For example, if we start with $\frac{2}{3}$, the following fractions are all equivalent:

$\frac{2}{3}=\frac{4}{6}$	$\frac{2}{3}=\frac{6}{9}$	$\frac{2}{3}=\frac{8}{12}$	$\frac{2}{3}=\frac{10}{15}$
multiply both numerator and denominator by 2	multiply both numerator and denominator by 3	multiply both numerator and denominator by 4	multiply both numerator and denominator by 5

Theorem: (also called the Fundamental Property of Fractions) Given a fraction $\frac{a}{b}$ where a and b are integers, $b \neq 0$, and any non-zero number c, $\frac{a}{b}$ and $\frac{ac}{bc}$ are equivalent fractions, i.e.

$$\frac{a}{b}=\frac{ac}{bc}$$

Another way to express this is as follows: One thing we can always do to a fraction without changing its value is to multiply both numerator and denominator by the same number. Equivalent fractions are essentially different representations of the same number.

Example 1. Re-write $\frac{4}{5}$ with a denominator of 20.

Solution: The only thing we can do is to multiply both numerator and denominator by the same number. If we want a new denominator of 20, that means that we have to multiply the old denominator, 5, by 4. Then we have no other option than multiplying the numerator by the same number. $4 \cdot 4 = 16$ and $5 \cdot 4 = 20$. Thus the equivalent fraction we are looking for is $\boxed{\frac{16}{20}}$.

Example 2. Re-write $\frac{4}{5}$ as a percent.

Solution: The only thing we can do is to multiply both numerator and denominator by the same number. Since we must write $\frac{4}{5}$ as a percent, that measn that we want a new denominator of 100. What number should we multiply by 5 to get to 100? This is a division problem, $100 \div 5 = 20$. This means that we will multiply both numerator and denominator by 20.

$$\frac{4}{5} = \frac{4 \cdot 20}{5 \cdot 20} = \frac{80}{100}$$

and $\frac{80}{100}$ is the same as $\boxed{80\%}$.

Now that we know that each fraction can be represented in infinitely many forms, we should find an "official" one, hopefully the simplest.

We now know that a fraction's value does not change if we multiply both numerator and denominator by the same non-zero number.

$$\frac{a}{b} = \frac{ac}{bc} \qquad b, c \neq 0$$

Reading the same equality from right to left as $\frac{ac}{bc} = \frac{a}{b}$ tells us that we are also allowed to *divide* both numerator and denominator by the same non-zero number, and we did not change the value of the fraction. This gives us the concept of the simplest form of a fraction.

Definition: Given a fraction $\frac{a}{b}$ where a and b are integers, $b \neq 0$, the fraction is **reduced** or **in lowest terms** if a and b have no common divisor greater than 1. If this is the case, we also say that a and b are **relatively prime**.

It feels true that every fraction has a unique reduced form. Since every fraction can be reduced, and the reduced form is clearly its simplest form, **we always present fractions in lowest terms as final answers**.

Example 3. Reduce $\frac{12}{28}$ to lowest terms.

Solution: We need to look for a common factor between 12 and 28. The greatest common divisor is 4. Indeed, when we divide both numerator and denominator by 4, we get that $\frac{12}{28} = \frac{3}{7}$.

Note that we do not have to get it right for the first time. Suppose we only notice the common factor 2. Then we have $\frac{12}{28} = \frac{6}{14}$. This is not reduced yet, because both 6 and 12 are even. So we go again: we divide both both numerator and denominator by 2 again: $\frac{12}{28} = \frac{6}{14} = \frac{3}{7}$. Now the fraction is in lowest terms, so the answer is $\boxed{\frac{3}{7}}$.

Example 4. Re-write 35% as a fraction in lowest terms.

Solution: First, $35\% = \frac{35}{100}$. Clearly both numerator and denominator are divisible by 5, so let's get rid of that 5 by dividing both numerator and denominator by 5. $\frac{35}{100} = \frac{7}{20}$. This is now in lowest form since 7 and 20 do not share any divisor besides the obvious one, 1. So 35% as a reduced fraction is $\boxed{\frac{7}{20}}$.

Until now, we only looked at fractions that were a part of a whole. If we cut a pizza into six equal slices and take all six slices, how can we express this as a fraction? Based on what we learned so far, $\frac{6}{6}$ is the fraction expressing this. Dividing both numerator and denominator by 6, we get $\frac{6}{6}=\frac{1}{1}$ or simply 1. 1 whole pizza.

So, the reduced form of fractions such as $\frac{3}{3}$ or $\frac{7}{7}$ or $\frac{12}{12}$ is simply 1.

What if we buy two pizzas, we slice them both into 6 equal slices and take 10 such slices? That could be expressed as $\frac{10}{6}$. And if we take all 12 slices? That would be $\frac{12}{6}=\frac{2}{1}=2$, or two 2 whole pizzas.

This means that every positive integer can be expressed as a fraction, with infinitely many equivalent forms. For example, $5=\frac{5}{1}=\frac{30}{6}$. Indeed, if we buy 5 whole pizzas and cut each into 6 equal slices, we will have 30 slices, each so big so that 6 of them makes an entire pizza.

Definition: If the numerator of a fraction is less than its denominator, we call such a fraction **proper**. A proper fraction always expresses less than a unit of a given quantity. Examples of proper fractions are $\frac{2}{3}$, $\frac{3}{10}$, $\frac{6}{15}$, or 55%.

If the numerator of a fraction is greater than or equal to its denominator, we call the fraction **improper**. An improper fraction always expresses a whole unit or more of a quantity. Examples of improper fractions are $\frac{5}{5}$, $\frac{12}{10}$, $\frac{40}{2}$, or 120%.

In spite of the suggested value judgement, we don't see anything improper about an improper fraction. Improper fractions can be presented as final values, as long as they are in lowest terms.

Example 5. Reduce $\frac{30}{12}$ to lowest terms.

Solution: Both 30 and 12 are divisible by 6. So we divide both numerator and denominator by 6 and we obtain that $\frac{30}{12}=\frac{5}{2}$. Since 5 and 2 are obviously relatively primes (i.e. share no divisor greater than 1), this is the final answer. So the reduced form of $\frac{30}{12}$ is $\boxed{\frac{5}{2}}$.

Example 6. Re-write 160% as a fraction in lowest terms.

Solution: $160\%=\frac{160}{100}$. We can easily divide both numerator and denominator by 10, that is just chopping off the last zero. $160\%=\frac{160}{100}=\frac{16}{10}$. This is still not reduced because both 16 and 10 are even. So we divide both numerator and denominator by 2 and get $160\%=\frac{160}{100}=\frac{16}{10}=\frac{8}{5}$. Since 5 and 8 are obviously relatively primes (i.e. share no divisor greater than 1), this is the final answer. The reduced fraction form of 160% is $\boxed{\frac{8}{5}}$.

Consider now the expression $\frac{15}{5}$. Are we looking at two integers with an operation, namely division between them, or are we looking at a single fraction, expressing 15 slices, where the slices are so big that five of them makes up a whole? There is a tremendeous duality here: we can read the same expression in two fundamentally different way. This is only allowed in mathematics if the duality never causes conflict or confusion. That is, every question has the same answer, no matter which of the two interpretations are we using. Instead of confusion, this duality is really a source of power. We can always look at a fraction as a division between integers, and we can always look at a division between integers as if it was a fraction.

Before the time of calculators, people used a lot of mental techniques to help in computation. The following is just an example of such a technique using the duality described above.

Example 7. Perform the division $\frac{140}{5}$ without the help of a calculator.

Solution: Even though this is a division between two integers, we will look at it for a while as if it was a fraction. If so, we are allowed to multiply numerator and denominator by the same number, and our chosen number is 2. This is because division by 10 is very easy: just chop off a zero at the end.

$$\frac{140}{5} = \frac{140 \cdot 2}{5 \cdot 2} = \frac{280}{10} = 28$$

Practice Problems

1. a) Re-write $\frac{2}{9}$ with a denominator of 45, i.e. find x so that $\frac{2}{9} = \frac{x}{45}$.

   b) Verify your result by taking $\frac{2}{9}$ of 90 and $\frac{x}{45}$ of 90. If you get the same result, your answer is probably correct.

2. a) Re-write $\frac{5}{6}$ with a numerator of 30.

   b) Re-write $\frac{5}{6}$ with a denominator of 30.

3. Re-write each of the given percents as a reduced fraction.

   a) 60% b) 25% c) 100% d) 50% e) 150% f) 220%

4. Re-write each of the given fractions as a percent.

   a) $\frac{3}{10}$ b) $\frac{17}{20}$ c) $\frac{60}{25}$ d) $\frac{3}{50}$ e) 5

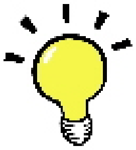
5. Use the duality between division of integers and standalone fractions to perform the given divisions without the aid of a calculator.

   a) $\frac{120}{5}$ b) $\frac{2050}{50}$ c) $\frac{1200}{25}$

6. a) Bring the fractions $\frac{7}{10}$ and $\frac{2}{3}$ to a common denominator. Compare the numerators.

   Which fraction is greater, $\frac{7}{10}$ or $\frac{2}{3}$?

   b) Compute both $\frac{7}{10}$ and $\frac{2}{3}$ of 60. Is your answer in conflict with part a)?



Answers

1. a) $\frac{2}{9} = \frac{10}{45}$ b) $\frac{2}{9}$ of 90 is 20, and so is $\frac{10}{45}$ of 90.

2. a) $\frac{30}{36}$ b) $\frac{25}{30}$

3. a) $\frac{3}{5}$ b) $\frac{1}{4}$ c) 1 d) $\frac{1}{2}$ e) $\frac{3}{2}$ f) $\frac{11}{5}$

4. a) 30% b) 85% c) 240% d) 6% e) 500%

5. a) 24 b) 41 c) 48

6. a) $\frac{7}{10} = \frac{21}{30}$ and $\frac{2}{3} = \frac{20}{30}$. Clearly, $\frac{21}{30} > \frac{20}{30}$ and so $\frac{7}{10} > \frac{2}{3}$.
   b) $\frac{7}{10}$ of 60 is 42, and $\frac{2}{3}$ of 60 is 40. Since $42 > 40$, we get again that $\frac{7}{10}$ is greater.

