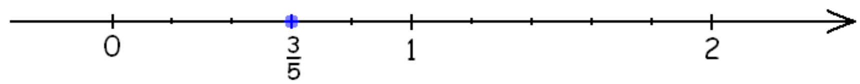
Part 1 - Fractions as Part of Our Number System

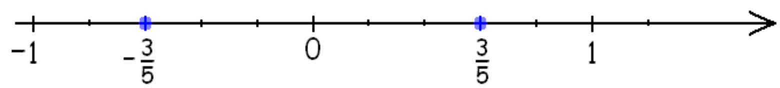
Until now, we have only looked at fractions as expressing a part of a quantity. Before we define the four basic operations on fractions, it is best to start looking at them as part of our number system.

We have defined what it means $\frac{3}{5}$ of somthing. What about $\frac{3}{5}$ alone, as a number? We understand $\frac{3}{5}$ alone as $\frac{3}{5}$ of the number 1. So, we can actually place fractions on the number line. In case of $\frac{3}{5}$, we divide the unit between zero and one into five equal parts, and place $\frac{3}{5}$ after three such line segments.



We can also interpret all integers as fractions. For example, 4 is the same as $\frac{8}{2}$ or $\frac{4}{1}$. The expression $\frac{4}{1}$ could mean division between the integers 4 and 1, and also, the fraction with numerator 4 and denominator 1. Thus duality is fundamental, and only allowed because all answers are always the same when worked with the two different meaning.

How about negative numbers? Negative fractions? We can clearly think if $-\frac{3}{5}$ as the opposite of $\frac{3}{5}$.



Definition: A *simplified* or *reduced* fraction cannot have more than one negative sign.

Let us consider the fraction $\frac{-2}{-7}$. Recall the fundamental property of fractions: that we can multiply both numerator and denominator by the same non-zero number without changing the value of the fraction. Let us proceed with -1.

$$\frac{-2}{-7} = \frac{-2(-1)}{-7(-1)} = \frac{2}{7}$$

So, a fraction with a negative number in the numerator and denominator can be simplified, and the result is simply a positive fraction. Notice that the rule is the same as with division between two integers.

The negative sign of a fraction can be located in three places. All three of the fractions shown below are equivalent.

$$-\frac{2}{3} \qquad \text{or} \qquad \frac{-2}{3} \qquad \text{or} \qquad \frac{2}{-3}$$

The first form, $-\frac{2}{3}$ $\left(\text{the opposite of } \frac{2}{3}\right)$ is considered simplified, and it is an elegant way to present a fraction as final answer. However, it is not the recommended form for computing with signed fractions.

The second form, $\frac{-2}{3}$ (the numerator is -2, the denominator is 3) is both simplified and highly recommended for computations. Indeed, when performing operations with signed fractions, we recommend that the negative sign is interpreted as belonging to the numerator.

The third form, $\dfrac{2}{-3}$ (the numerator is 2, the denominator is -3) is not considered simplified, and not recommended for computations either. Thus we recommend that when faced with such a fraction, we immediately multiply both numerator and denominator by -1, thereby switching to the second form, $\dfrac{-2}{3}$. **Fractions with a negative sign in their denominators are not acceptable as final answers.**

Example 1. Simplify each of the given fractions.

a) $\dfrac{10}{-28}$ b) $\dfrac{-12}{1}$ c) $\dfrac{-3}{-18}$ d) $\dfrac{4}{-7}$

Solution: a) The fraction itself can be simplified by dividing both numerator and denominator by 2. But then $\dfrac{5}{-14}$ is not simplified because the negative sign is in the wrong place. Multiplying both numerator and denominator by -1 will give us the answer in the correct form, $\dfrac{-5}{14}$. We can also present this as $-\dfrac{5}{14}$.

$$\frac{10}{-28} = \frac{5}{-14} = \boxed{\frac{-5}{14} \text{ or } -\frac{5}{14}}$$

Eventually we will be able to speed up the process by dividing numerator and denominator of $\dfrac{10}{-28}$ by -2.

b) The fraction $\dfrac{-12}{1}$ is not simplified, because it can be simplified to -12. Integers should never be presented as fractions. Clearly -12 is simpler than $\dfrac{-12}{1}$. So, the correct answer is $\boxed{-12}$.

c) We can simplify $\dfrac{-3}{-18}$ by dividing numerator and denominator by -3.

$$\frac{-3}{-18} = \boxed{\frac{1}{6}}$$

d) The fraction $\dfrac{4}{-7}$ is noot simplified only because the negative sign is in the denominator. We can fix that by multplying numerator and denominator by -1.

$$\frac{4}{-7} = \frac{4\cdot(-1)}{-7\cdot(-1)} = \boxed{\frac{-4}{7} \text{ or } -\frac{4}{7}}$$

Example 2. Re-write the mixed number $-4\dfrac{5}{8}$ as an improper fraction.

Solution: We simply ignore the negative sign, convert the mixed number to an improper fraction, and then put the negative sign back.

$$-4\frac{5}{8} = -\left(\frac{4\cdot 8+5}{8}\right) = -\frac{37}{8}$$

So the correct answer is $\boxed{-\dfrac{37}{8} \text{ or } \dfrac{-37}{8}}$.

Part 2 - Adding and Subtracting Fractions with the Same Denominator

Recall that the numerator and denominator play very different roles. The numerator counts how many slices we have. The denominator only describes how large each slice is. For example, in the fraction $\frac{3}{8}$, we are dealing with three slices. Each slice is so large so that eight of them makes up a whole. So, $\frac{3}{8}$ and $\frac{3}{5}$ both express three slices, only the slices in $\frac{3}{5}$ are bigger that those in $\frac{3}{8}$.

To add or subtract fractions with the same denominator, we write a single denominator and express the addition or subtraction of the signed numbers in the numerator.

For example, the addition $\frac{2}{7}+\frac{3}{7}$ is simple: we are talking about the same size of slices, and we add three slices to two slices. Thus $\frac{2}{7}+\frac{3}{7}=\frac{2+3}{7}=\frac{5}{7}$.

Consider now the addition $\frac{4}{5}+\frac{2}{5}$. Again, the same denominator indicates that the slices are of the same size, so we simpliy add two slices to the four. In short, $\frac{4}{5}+\frac{2}{5}=\frac{4+2}{5}=\frac{6}{5}$.

The answer is an improper fraction. However, we do not see anything imporper about it, as long as it is in lowest terms. We could present the final answer as $1\frac{1}{5}$ but $\frac{6}{5}$ is correct, simplified, and is actually preferred.

In case of signed numbers, the negative sign should always be kept in the numerator. Then the addition or subtraction of signed fractions will be reduced to addition or subtraction of (signed) integers in the numerator.

Example 3. Perform the addition or subtraction of the fractions as indicated.

a) $\frac{3}{8}-\frac{7}{8}$ b) $\frac{2}{5}-\frac{4}{5}-\left(-\frac{3}{5}\right)$ c) $\frac{3}{7}-\frac{9}{7}$ d) $\frac{3}{10}-\frac{7}{10}+\left(-\frac{1}{10}\right)$ e) $2\frac{1}{3}-5\frac{2}{3}$

Solution: a) We re-write the subtraction as one in the numerator. Then, if we can, we simplify the answer.

$$\frac{3}{8}-\frac{7}{8}=\frac{3-7}{8}=\frac{-4}{8}=\boxed{\frac{-1}{2}\text{ or }-\frac{1}{2}}$$

b) We re-write the subtraction with one big fraction bar and a subtraction of integers the same numerator. Then, if we can, we simplify the answer.

$$\frac{2}{5}-\frac{4}{5}-\left(-\frac{3}{5}\right)=\frac{2-4-(-3)}{5}=\frac{-2-(-3)}{5}=\frac{-2+3}{5}=\boxed{\frac{1}{5}}$$

c) We re-write the subtraction with one big fraction bar and a subtraction of integers the same numerator. Then, if we can, we simplify the answer.

$$\frac{3}{7}-\frac{9}{7}=\frac{3-9}{7}=\boxed{\frac{-6}{7}\text{ or }-\frac{6}{7}}$$

d) We re-write the subtraction with one big fraction bar and a subtraction of integers the same numerator. Then, if we can, we simplify the answer.

$$\frac{3}{10}-\frac{7}{10}+\left(-\frac{1}{10}\right)=\frac{3-7+(-1)}{10}=\frac{-4+(-1)}{10}=\frac{-5}{10}=\boxed{\frac{-1}{2}\text{ or }-\frac{1}{2}}$$

e) In case of adding or subtracting mixed numbers, we recommend the use of improper fractions.

$$2\frac{1}{3}-5\frac{2}{3}=\frac{7}{3}-\frac{17}{3}=\frac{7-17}{3}=\boxed{\frac{-10}{3}\text{ or }-\frac{10}{3}}$$

There are other ways of handling this, but we recommend the method shown above. However, there is a way to do this subtraction by separating integer and fraction parts. We need to be careful though to correctly interpret the negative signs and subtraction signs to both integer and fraction parts.

$$2\frac{1}{3}-5\frac{2}{3}=\underbrace{2-5}_{\text{integer part}}+\underbrace{\frac{1}{3}-\frac{2}{3}}_{\text{fraction part}}=-3+\left(-\frac{1}{3}\right)=-3\frac{1}{3}$$

We got lucky with the last problem because both integer and fraction part ended up with the same sign in the result. What about a subtraction such as $4\frac{2}{5}-1\frac{4}{5}$?

$$4\frac{2}{5}-1\frac{4}{5}=\underbrace{4-1}_{\text{integer part}}+\underbrace{\frac{2}{5}-\frac{4}{5}}_{\text{fraction part}}=3+\left(-\frac{2}{5}\right)$$

We can not present a mixed number with mixed signs. The sign of the entire mixed number is determined by the integer part, because it has more value than the fraction part. This answer should be positive. How do we express 3 and $-\frac{2}{5}$ as a mixed number? We borrow 1 from the integer part, express it as $\frac{5}{5}$ and throw it in with the integer part.

$$3+\left(-\frac{2}{5}\right)=2+1+\left(-\frac{2}{5}\right)=2+\frac{5}{5}+\left(-\frac{2}{5}\right)=2+\frac{5+(-2)}{5}=2+\frac{3}{5}=2\frac{3}{5}$$

This can get quite complicated, which is why we recommend to use improper fractions instead of mixed numbers.

$$4\frac{2}{5}-1\frac{4}{5}=\frac{22}{5}-\frac{9}{5}=\frac{22-9}{5}=\frac{13}{5}\qquad\left(\text{same as }2\frac{3}{5}\right)$$

Part 3 - Adding and Subtracting Fractions with Different Denominators

How do we add the fractions $\frac{5}{6}$ and $\frac{3}{4}$? It is true that we are adding five slices and three slices, but they do not have the same size. So, how do we add fractions with different denominators? Well, we just don't. Instead, we use the fundamental property of fractions to bring the fractions to a common denominator.

The fundamental property of fractions is that we can multiply both numerator and denominator by the same non-zero number without changing the value of the fraction. Therefore, we can find fractions equivalent to $\frac{5}{6}$ with all denominators that are multiples of 6. For example, $\frac{5}{6}=\frac{10}{12}=\frac{15}{18}=\frac{20}{24}=\frac{30}{36}=\frac{50}{60}$ and so on. Similarly, we can find fractions equivalent to $\frac{3}{4}$ with all denominators that are multiples of 4. For example, $\frac{3}{4}=\frac{6}{8}=\frac{9}{12}=\frac{12}{20}=\frac{15}{20}=\frac{18}{24}$, and so on.

If we multiply 6 and 4, we get a common multiple, 24. However, we should always attempt to work with **the least common multiple**, and that is 12 in this case. To bring $\frac{5}{6}$ to a denominator 12, we multiply numerator and denominator by 2. To bring $\frac{3}{4}$ to a denominator 12, we multiply numerator and denominator by 3.

$$\frac{5}{6}=\frac{5\cdot 2}{6\cdot 2}=\frac{10}{12}\quad\text{and}\quad\frac{3}{4}=\frac{3\cdot 3}{4\cdot 3}=\frac{9}{12}\quad\text{Thus we have that}\quad\frac{5}{6}+\frac{3}{4}=\frac{10}{12}+\frac{9}{12}=\frac{10+9}{12}=\frac{19}{12}$$

This method also works with adding fractions with integers, because we can always re-write an integer as a fraction with denominator 1.

Example 4. Perform the addition or subtraction of the fractions as indicated.

a) $\dfrac{3}{10}-\dfrac{4}{5}$ b) $4-\dfrac{2}{3}-\left(-\dfrac{1}{2}\right)$ c) $\dfrac{2}{5}-3$ d) $\dfrac{2}{3}-\dfrac{5}{6}-\left(-\dfrac{1}{2}\right)$ e) $4\dfrac{1}{6}-3\dfrac{2}{3}$

Solution: a) The least common denominator between 10 and 5 is 10. We re-write $\dfrac{4}{5}$ as a fraction with denominator 10. by multiplying both numerator and denominator by 2. $\dfrac{4}{5}=\dfrac{4\cdot 2}{5\cdot 2}=\dfrac{8}{10}$

$$\frac{3}{10}-\frac{4}{5}=\frac{3}{10}-\frac{8}{10}=\frac{3-8}{10}=\frac{-5}{10}=\boxed{\frac{-1}{2}\text{ or }-\frac{1}{2}}$$

b) The integer 4 is not a fraction, but that can be easily fixed, by re-writing it as $\dfrac{4}{1}$. We are now looking for the least common multiple of 1, 3, and 2. Clearly, 6 will be good as the least common multiple. Then $\dfrac{4}{1}$ can be re-written as $\dfrac{4}{1}=\dfrac{4\cdot 6}{1\cdot 6}=\dfrac{24}{6}$. Similarly, $\dfrac{2}{3}=\dfrac{2\cdot 2}{3\cdot 2}=\dfrac{4}{6}$ and $\dfrac{-1}{2}=\dfrac{-1\cdot 3}{2\cdot 3}=\dfrac{-3}{6}$. Once the fractions have the same denominator, we perform the additions and subtractions in the numerator. If there is a negative sign, we keep it in the numerator.

$$4-\frac{2}{3}-\left(-\frac{1}{2}\right)=\frac{4}{1}-\frac{2}{3}-\left(\frac{-1}{2}\right)=\frac{24}{6}-\frac{4}{6}-\left(\frac{-3}{6}\right)=\frac{24-4-(-3)}{6}=\frac{20-(-3)}{6}=\boxed{\frac{23}{6}}$$

c) We re-write 3 as $\dfrac{3}{1}$. The least common multiple of 1 and 5 is 5. To re-write $\dfrac{3}{1}$ with a denominator 5, we multiply both numerator and denominator by 5. $\dfrac{3}{1}=\dfrac{3\cdot 5}{1\cdot 5}=\dfrac{15}{5}$

$$\frac{2}{5}-3=\frac{2}{5}-\frac{3}{1}=\frac{2}{5}-\frac{15}{5}=\frac{2-15}{5}=\boxed{\frac{-13}{5}\text{ or }-\frac{13}{5}}$$

We can express the answer as a mixed number, but there is no need, unless we are specifically asked for it.

d) The least common multiple of 3, 6, and 2 is 6. So we will re-write each fraction with a denominator of 6. If there is a negative sign, we will keep it in the numerator.

$$\text{So, }\frac{2}{3}=\frac{2\cdot 2}{3\cdot 2}=\frac{4}{6}\quad\text{and}\quad\frac{5}{6}=\frac{5}{6}\quad\text{and}\quad-\frac{1}{2}=\frac{-1}{2}=\frac{-1\cdot 3}{2\cdot 3}=\frac{-3}{6}$$

We can now perform the subtractions, left to right.

$$\frac{2}{3}-\frac{5}{6}-\left(-\frac{1}{2}\right)=\frac{4}{6}-\frac{5}{6}-\left(\frac{-3}{6}\right)=\frac{4-5-(-3)}{6}=\frac{-1-(-3)}{6}=\frac{-1+3}{6}=\frac{2}{6}=\boxed{\frac{1}{3}}$$

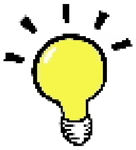
e) In case of mixed numbers, we again recommend to switch to improper fractions. We re-write $4\dfrac{1}{6}$ as $\dfrac{25}{6}$. Notice that the addition $4+\dfrac{1}{6}$ gives us exactly the same result. $3\dfrac{2}{3}$ can be re-written as $\dfrac{11}{3}$. The least common denominator between 6 and 3 is 6.

$$4\frac{1}{6}-3\frac{2}{3}=\frac{25}{6}-\frac{11}{3}=\frac{25}{6}-\frac{22}{6}=\frac{25-22}{6}=\frac{3}{6}=\boxed{\frac{1}{2}}$$

Practice Problems

Perform the indicated operations. Present your answer as an integer or a reduced fraction. You do not need to convert improper numbers to mixed numbers.

1. $\frac{2}{3}+\frac{5}{6}$
2. $\frac{2}{3}-\frac{5}{6}$
3. $-2-\frac{2}{3}$
4. $-\frac{2}{5}-\left(-\frac{5}{2}\right)$
5. $\frac{3}{4}-\left(-\frac{5}{8}\right)$
6. $\frac{5}{4}-\frac{1}{6}$
7. $2-\frac{2}{3}$
8. $-2+\frac{5}{8}$
9. $2\frac{4}{5}-3\frac{1}{2}$
10. $-3\frac{1}{4}-2\frac{5}{6}$
11. $\frac{1}{3}-\frac{3}{4}+\frac{5}{12}$
12. $-\frac{3}{4}-\frac{5}{6}$
13. $\frac{5}{12}-\frac{3}{4}$
14. $\frac{19}{3}-2\frac{5}{6}$
15. $\frac{5}{2}-\frac{7}{10}+\frac{6}{5}$



Answers

1. $\frac{3}{2}$ 2. $-\frac{1}{6}$ 3. $-\frac{8}{3}$ 4. $\frac{21}{10}$ 5. $\frac{11}{8}$ 6. $\frac{13}{12}$ 7. $\frac{4}{3}$ 8. $-\frac{11}{8}$ 9. $-\frac{7}{10}$ 10. $-\frac{73}{12}$

11. 0 12. $-\frac{19}{12}$ 13. $-\frac{1}{3}$ 14. $\frac{7}{2}$ 15. 3