Part 1: Converting a Fraction to a Decimal

This is easy to do if we understand a formal, algebraic definition of a fraction. If a and b are integers, b not zero, then the **fraction** $\frac{a}{b}$ is the result of the division $a \div b$. In a sense, fractions are driving instructions. They do not tell us the value of the number, only, how to obtain it. To get a decimal, we simply perform the division.

Example 1 Convert $\frac{3}{8}$ to a decimal.

Solution: We perform the division $3 \div 8$. The result is 0.375.

		.3	7	5
8	)3	.0	0	0
−	2	4		
		6	0	
	−	5	6	
			4	0
		−	4	0
				0

Example 2 Convert $\frac{1927}{11}$ to a decimal.

We perform the division $1927 \div 11$. The result is $175.\overline{18}$. The bar over the last two digits indicates an infinitely many times repeating block.

			1	7	5	.1	8	1	8...
11	)	1	9	2	7	.0	0	0	0
−		1	1						
			8	2					
		−	7	7					
				5	7				
			−	5	5				
					2	0			
				−	1	1			
						9	0		
					−	8	8		
							2	0	
						−	1	1	
								9	0
							−	8	8
									2

Part 2: Converting a Terminating Decimal to Fraction

A decimal is **terminating** if it has a last digit. It is quite easy to turn a terminating decimal to a fraction of integers.

Example 3 Convert 0.45 to a reduced fraction.

Step 1. Write it as a fraction of any kind first.

$$0.45 = \frac{0.45}{1}$$

We can mentally check it as division: any number divided by one results in the same number.

Step 2. We ask ourselves: "How many digits do we need to move the decimal point to the right in 0.45 to obtain an integer"? The answer is: two digits. Moving the decimal point to the right by two digits is the same as multiplication by 100. Thus, to fix the numerator, we need to multiply it by 100. Because we also want to preserve the value, we multiply both upstairs and downstairs by 100.

$$\frac{0.45}{1} = \frac{0.45 \cdot 100}{1 \cdot 100} = \frac{45}{100}$$

Step 3. We simplify the fraction by dividing upstairs and downstairs by the greatest common divisor.

$$\frac{45}{100} = \frac{\not{5} \cdot 9}{\not{5} \cdot 20} = \frac{9}{20}$$

Example 4 Convert 0.0005 to a reduced fraction.

We will multiply upstairs and downstairs by 10000.

$$0.0005 = \frac{0.0005}{1} = \frac{0.0005 \cdot 10000}{1 \cdot 10000} = \frac{5}{10000} = \frac{\not{5} \cdot 1}{\not{5} \cdot 2000} = \frac{1}{2000}$$

Example 5 Convert 23.044 to a reduced fraction.

$$23.044 = 23 + 0.044 = 23 + \frac{0.044}{1} = 23 + \frac{0.044 \cdot 1000}{1 \cdot 1000} = 23\frac{44}{1000} = 23\frac{4 \cdot 11}{4 \cdot 250} = 23\frac{11}{250}$$

Part 3: (The Fun Stuff)
Converting a Non-Terminating Decimal to Fraction

A decimal is non-terminating if it has infinitely many digits. If there is a repeating block, we denote it by a bar drawn over the repeating digit. For example, the number $2.\overline{35}$ denotes 2.35 35 35 35......

Example 6 Re-write each of the given repeating decimals without the bar notation.

a) $1.201\overline{7}$ b) $1.20\overline{17}$ c) $1.2\overline{017}$ d) $1.\overline{2017}$

Solution: Only the digit(s) under the bar are repeating. The rest is there as is.

a) $1.201\overline{7} = 1.201\,777777....$

b) $1.20\overline{17} = 1.20\,17\,17\,17\,17\,17\,17....$

c) $1.2\overline{017} = 1.2\,017\,017\,017\,017\,017....$

d) $1.\overline{2017} = 1.2017\,2017\,2017\,2017\,2017....$

Turning these decimals into fractions of integers is an interesting and fun application of linear equations.

Example 7 Convert the repeating decimal $7.\overline{4}$ to a fraction.

Step 1. We label our number x and write it without the bar notation. The dots are important: they indicate that we have infinitely many $4'$s there and not just three.

$$7.444... = x$$

Step 2. We multiply both sides of this equation by 10.

$$74.444... = 10x$$

Step 3. We write these equations together, starting with the second one.

$$\begin{aligned} 74.444... &= 10x \\ 7.444... &= x \end{aligned}$$

Step 3. (Chop, chop.) We subtract the second equation from the first one.

$$\begin{array}{rl} & 74.444... = 10x \\ - & \underline{7.444... = x} \\ & 67 \qquad\quad = 9x \end{array}$$

Step 4. We solve the equation for x.

$$\begin{aligned} 67 &= 9x \qquad \text{divide by } 9 \\ \frac{67}{9} &= x \end{aligned}$$

Thus the answer is $\frac{67}{9}$. We can check by long division. Indeed, $67 \div 9 = 7.444\,444\,44...$

Example 8 Convert the repeating decimal $0.\overline{405}$ to a fraction.

Step 1. We label our number x and write it without the bar notation. $0.405\,405\,405\,405\,405... = x$

Steps 2 and 3. This decimal has a three-digit long repeating block. To obtain proper alignment of the digits, we will move the decimal point by three digits, i.e. we will multiply by 1000. We multipliy both sides of this equation by 1000. We write these equations together, starting with the second one.

$$\begin{aligned} 405.405\,405\,405\,405... &= 1000x \\ 0.405\,405\,405\,405... &= x \end{aligned}$$

Step 3. (Chop, chop.) We subtract the second equation from the first one.

$$\begin{array}{rl} & 405.405\,405\,405\,405... = 1000x \\ - & \underline{0.405\,405\,405\,405.... = x} \\ & 405 \qquad\qquad\qquad\qquad = 999x \end{array}$$

Step 4. We solve the equation for x.

$$\begin{aligned} 405 &= 999x \qquad \text{divide by } 999 \\ \frac{405}{999} &= x \end{aligned}$$

Please note that the fraction obtained is not reduced. However, the essential point in the problem is to find a fraction, not the reduced form of it. Thus the answer is $\frac{405}{999}$. We can check by long division. Indeed, $405 \div 999 = 0.405\,405\,405....$

Example 9 Convert the repeating decimal $18.29\overline{04}$ to a fraction.

Step 1. We label our number x and write it without the bar notation.

$$18.29\,04\,04\,04\,04\ldots = x$$

Steps 2 and 3. This decimal has a two-digit long repeating block. To obtain proper alignment of the digits, we will move the decimal point by two digits, i.e. we will multiply by 100. We multipliy both sides of this equation by 100. We write the two equations together, starting with the second one.

$$\begin{aligned} 1829.04\,04\,04\,04\,04\ldots &= 100x \\ 18.29\,04\,04\,04\,04\ldots &= x \end{aligned}$$

Step 3. (Chop, chop.) We subtract the second equation from the first one.

$$\begin{array}{rl} 1829.04\,04\,04\,04\,04\ldots &= 100x \\ -\quad 18.29\,04\,04\,04\,04\ldots &= x \\ \hline 1810.75 \qquad\qquad &= 99x \end{array}$$

It appears that we have a problem: the right-hand side is not an integer after the subtraction. This is quite easy to fix: we just multipliy both sides by 100.

$$\begin{aligned} 1810.75 &= 99x && \text{multiply by 100} \\ 181075 &= 9900x \end{aligned}$$

Step 4. We solve the equation for x.

$$\begin{aligned} 181075 &= 9900x && \text{divide by 9900} \\ \frac{181075}{9900} &= x \end{aligned}$$

Thus the answer is $\dfrac{181075}{9900}$. We can check by long division. Indeed, $181075 \div 9900 = 18.290\,40404\ldots$

Practice Problems

1. Perform each of the following conversions.

   a) Convert the given fraction to decimals.

   i) $\dfrac{4}{5}$ ii) $\dfrac{26}{3}$ iii) $\dfrac{26}{25}$ iv) $\dfrac{26}{7}$

   How many digits long is the repeating block?

   b) Convert the given decimal to a fraction of integers. (You do not have to reduce them!)

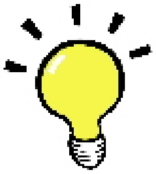
   i) 2.18 ii) $2.\overline{9}$ iii) $6.\overline{47}$ iv) $1.8\overline{705}$

2. Based on your answer for 1b ii), what is a surprising new fact about decimal presentation of numbers?

3. Consider the fraction $\dfrac{1}{n}$. What numbers n will result in a terminating decimal?

4. The decimal presentation of $\dfrac{2}{13}$ does not appear to be repeating. Is it?

5. Find a fraction formed of two integers that will result in a non-terminating, non-repeating decimal.



Answers to Practice Problems

1. a) i) 0.8 ii) $8.\overline{6}$ iii) 1.04 iv) $3.\overline{714285}$, the repeating block is 6 digits long

   b) i) $\dfrac{218}{100}$ ii) 3 iii) $\dfrac{641}{99}$ iv) $\dfrac{18687}{9990}$

2. The decimal presentation of real numbers is not unique. 3. n can only have 2 and 5 in its prime-factorization

4. It is repeating, only the repeating block is 6 digits long, $0.\overline{153846}$ 5. It is impossible