## Part 1- What is Mathematics?

As a graduate student, I had the annoying habbit of asking my teachers and peer students what they think mathematics is. To my surprise, I received many different answers, and to this day, I agree with many of them. In my eyes, mathematics is many things. In mathematics, we will be talking a lot about things being true or being not true. Although this probably happens during every course in every discipline, mathematical truth can be objectively established and agreed upon. To achieve such an objective approach, we have to develop a language that is objectively understood. In this sense, mathematics is also a language.

> **Mathematics** is a collection of *true statements* that are developed, expressed, and interpreted using an objective *language* and rules of *logic*.

To understand mathematics, we need to first agree on an objective language. Reading and writing mathematical notation correctly will be important.
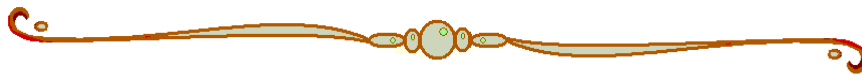
'Then you should say what you mean', the March Hare went on. 'I do,' Alice hastily replied; 'at least - at least I mean what I say - that's the same thing, you know.' 'Not the same thing a bit!' said the Hatter. Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see!'

Lewis Carroll
Alice's Adventures in Wonderland

Part 2 - Words and Sentences

Mathematical statements are much like English sentences. As English sentences are built from different kind of words, mathematical statements usually contain three types of components: **objects, operations and relations**.

> **Definition:** The concept of an **object** (very much like nouns in English sentences) is usually clearly understood and needs no explanation.

Examples of objects from algebra include $2, 3$, and generally, numbers. Objects from geometry include lines, points, triangles, circles, line segments, etc.

> **Definition:** An **operation** is an action (very much like verbs in English sentences) that can be applied to objects and usually result in new objects.

Examples of operations from algebra include addition, subtraction, multiplication, and division. Operations from geometry include reflection to a line, rotation, or translation that can be performed on points, triangles, circles, line segments, etc.

> **Definition:** A **relation** is something we use to compare two objects.

Relations, unlike operations do not produce new things, we use them to compare already existing objects. For example, the operation addition produces 7 if applied to 2 and 5. Relations in algebra are equal, or less, or greater. Relations from geometry are how geometric objects can be comapred to each other: similar, congruent, parallel, perpendicular.

It is actually pretty difficult to form a meaningful statement without using at least one of each of these three components. In the statement $2 + 5 = 7$ the numbers 2, 5, and 7 are the objects, addition (denoted by $+$) is the operation, and being equal (denoted by $=$) is the relation.

It is a common misconception to think of mathematics as the study of *only* numbers. Numbers are only certain types of objects. As we progress in the study of mathematics, we will find that there are many other types of interesting objects. For example, sets are objects we will soon study. Furthermore, the study of operations and relations is also interesting and fruitful.

So, what kind of are true statements can be established in mathematics? There are three types of true statements in mathematics, **definitions**, **axioms**, and **theorems**.

> **Definition:** A **definition** is a labeling statement in which we agree to use an expression to refer to an object, operation, or relation in mathematics.

Definitions are all true statements, because they simply reflect an agreement in the terms of the language. To be precise, these decisions were made without consultation with us, often decades (if not centuries) before we were even born.

An example for a definition would be if we pointed to a clear sky and said: 'From now on, let's call this color blue'.

> **Definition:** A **theorem** is a statement that we insist on proving before believing that it is true. To **prove** a theorem means to derive it from previously established true statements, using logically correct steps.

If you think about the last definition a little bit, you will see that no theorem can exist unless we agree on accepting a few statements to be true, without proving them. These are our "starting true statements".

> **Definition:** An **axiom** is a statement we agree to accept to be true without a proof.

Axioms are usually simple and basic statements that are in agreement with our intuition.

For example, the statement '*It is possible to draw a straight line from any point to any other point.*' is an axiom. It has been a constant effort to keep the number of axioms to a minimum. We prove a theorem by deriving its statement from statements already established to be true.

To be precise, when we prove our first theorem, we derive its statement from the axioms. When we prove our second theorem, we derive its statement from the axioms and the first theorem. When proving the third theorem, we can use all the axioms, and the first and second theorems. And so on. For our tenth theorem, we have all the axioms and the first nine theorems at our disposal. At this point, we are buliding a logically sound theory, a unified discipline within mathematics. It is one thing

to suspect, to feel, or to have a hunch that something is true. It is entirely different from proving it, with unescapable force of logic.

The ancient Greek mathematician Euclid discussed mathematics in such a manner, i.e stating axioms and building a theory by deriving a sequence of theorems from the axioms. (He called axioms postulates.) Mathematicians immediately accepted and embraced this logical approach to the study of mathematics - and it is how it is done still today. Although Euclid has contributed to several parts of mathematics (including geometry and number theory), he completely axiomatized of what we now call classical geometry or Euclidean geometry. He stated five postulates, accepted them to be true and derived most basic theorems of classical geometry.

Enrichment

1. Look up Euclid's Elements on the internet. (Start at Wikipedia). What is Elements? What is Euclid's contribution to **all** subjects within today's mathematics? What are postulates? List Euclid's five postulates. Explain the significance of these five statements.

2. What is the parallel postulate? Look up the history of Euclid's parallel postulate on the internet. (Start with Wikipedia.) How would we go about proving that an axiom is really a theorem? List statements from geometry that are logically equivalent to the parallel postulate. What exactly could it mean for two axioms to be equivalent to each other?

Euclid (~400 BC - ~300 BC)

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