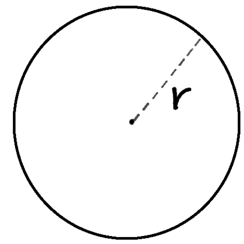
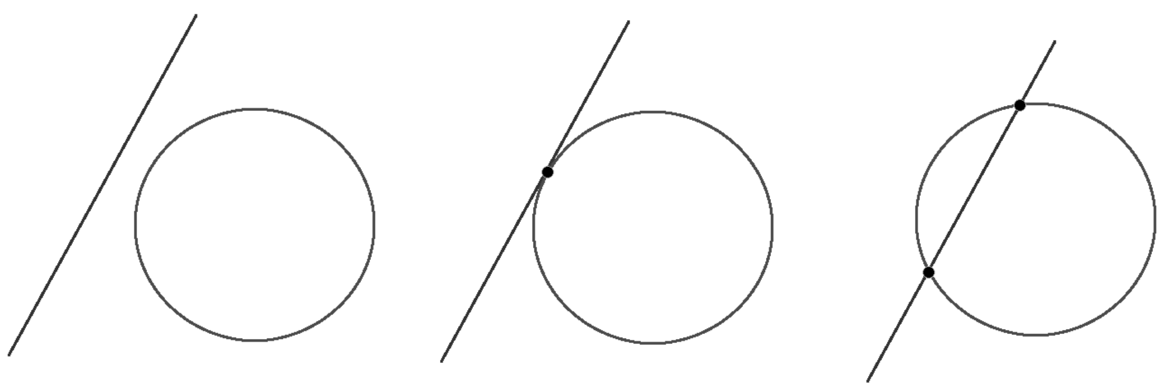
Recall the definition of a circle.

Definition: A circle is the set of all points in a plane that are equidistant to a fixed point. That equal distance is called the radius of the circle, that fixed point is called the center of the circle.



Part 1 - Tangent Lines

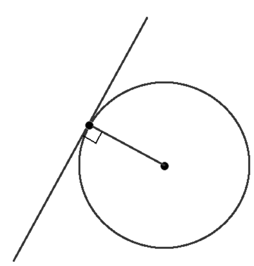
A circle and a line in the same plane could have no intersection points, one intersection point, or two intersection points. As it turns out, the case with the single intersection point is interesting and very important.



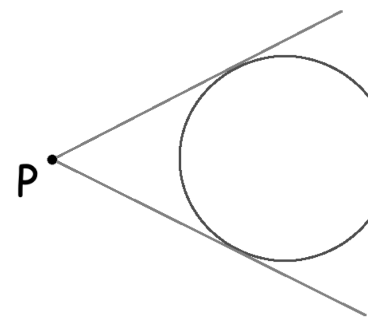
Definition: Suppose that a circle and a line in the same plane have exactly one intersection point. In this case, we say that the line is a **tangent line** of the circle. The point shared between the line and circle is called the **point of tangency**.

The following two theorems will be important. They often pop up in geometry proofs and problems.

Theorem: If a line is tangent to a circle, then the radius drawn to the point of tangency is perpendicular to the tangent line.



If a point P is outside of the circle, (we also say that P is an external point), then there are two tangent lines from P to the circle.

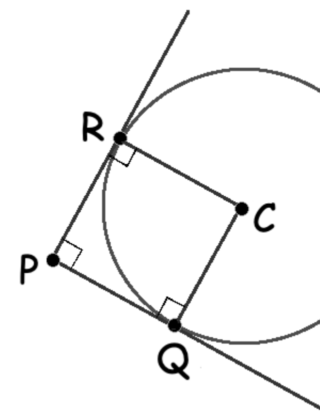


Example 1. Given a circle, find all points P such that the tangent lines drawn to the circle from P are perpendicular to each other.

Solution: Let P be a point so that the tangent lines through P are perpendicular to each other. Let us label the center of the circle as C and the radius of the circle by r. Let Q and R be the points of tangency.

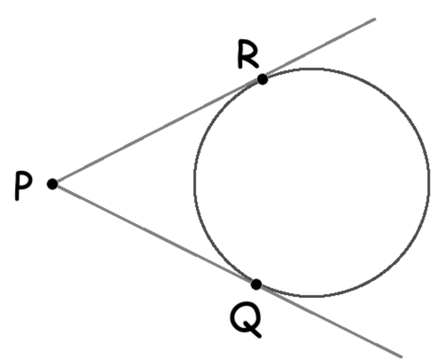
Consider now the quadrilateral $PQCR$. It is a rectangle because three of its angles are $90°$. Then the fourth angle is also a right angle, and so we have a rectangle. Furthermore, it is a square because RC and CQ are equally long because they are radii in the same circle. So all four sides are of length r.

By the Pythagorean theorem, the distance of P and C must be $\sqrt{2}\,r$.



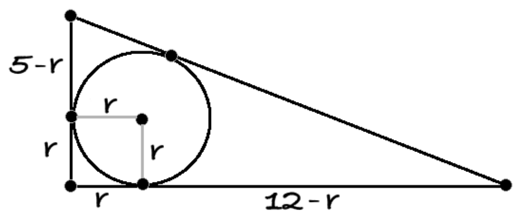
So the points P with the required property are all on a circle, centered at C, with radius $\sqrt{2}\,r$ where r is the radius of the circle.

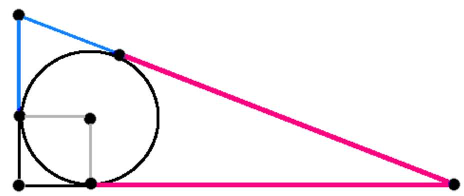
Theorem: (Tangent Segments Theorem) Consider P, a point external to a circle. Suppose we drew both tangent lines from P to the circle. If the points of tangency are Q and R, then $\overline{PQ} = \overline{PR}$.



Example 2. For every triangle, there exists a unique circle inside the triangle such that all three sides are tangents to the circle. Such a circle is called the inscribed circle. Consider the right triangle with sides 5, 12, and 13 unit long. Find the radius of its inscribed circle.

Solution: The vertex opposite the hypotenuse, the center of the circle and the points of tangency on the shorter sides form a rectangle because three of its angles measure $90°$. This rectangle is actually a square with sides r.





The point of tangency splits the shortest side into line segments r and $5-r$ long. The point of tangency splits the second longest side into line segments r and $12-r$ long.

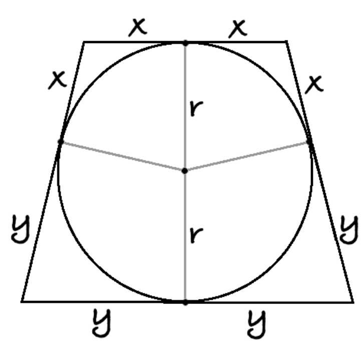
If two tangent lines have a common external point, then the tangent line segments are equally long. This enables us to write a surprisingly simple equation expressing the length of the hypotenuse.

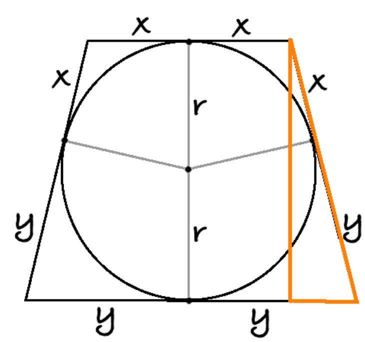
$$5-r+12-r=13$$

We solve the equation and get $r=2$.

Example 3. A circle is inscribed in an isosceles trapezoid, i.e. all four sides are tangent to the circle inside the trapezoid. Let d denote the diameter of the circle, and a and b the horizontal sides of the trapezoid. Prove that then $d=\sqrt{ab}$.

Solution: Let us denote the tangent line segments by x and y as shown on the picture. Consider now the triangle marked orange. The vertical side is $2r$, the horizontal side is $y-x$, and the hypotenuse is $y+x$.





We state the Pythagorean theorem for the triangle marked orange. Let d denote the diameter, i.e. $d=2r$ and also, the parallel sides are $a=2x$ and $b=2y$.

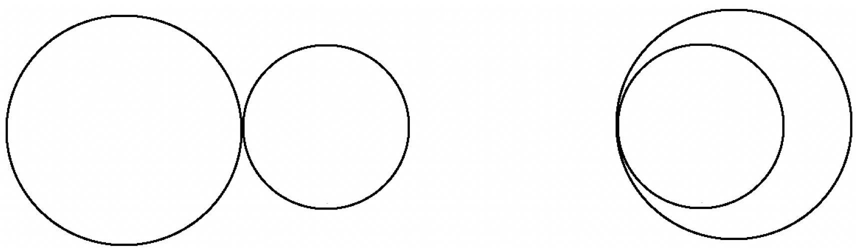
$$\begin{aligned}
d2+(y-x)^2 &= (y+x)^2 \\
d^2+x^2+y^2-2xy &= x^2+y^2+2xy \\
d^2 &= 4xy \\
d^2 &= (2x)(2y) \\
d^2 &= ab
\end{aligned}$$

Therefore, $d = \sqrt{ab}$ or $d = -sqrtab$. Since distances cannot be negative, $d = \sqrt{ab}$ is true which completes our proof.

Note that the relationship $d = \sqrt{ab}$ has a name: we say that d is the geometric mean of a and b. The name is not exactly a coincidence: this relationship shows up often in geometry.

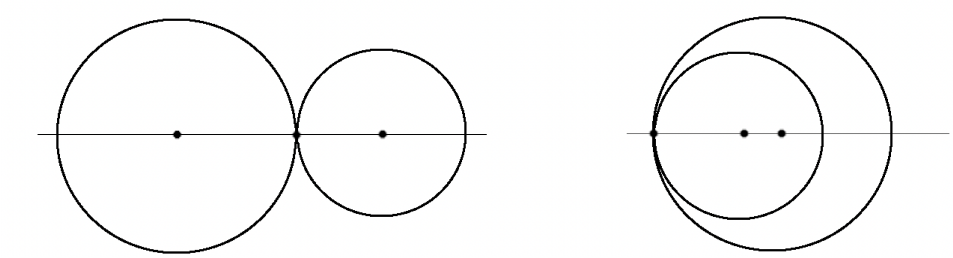
Part 2 - Circles Tangent to Each Other

Circles can also be tangent to each other. They can be internally or externally tangent.

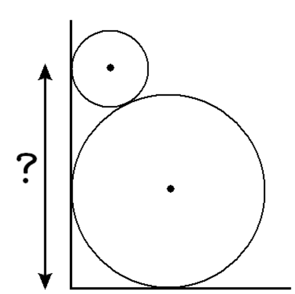


Theorem: If two circles are tangents to each other (externally or internally), then the line connecting the centers will pass through the point of tangency.

This fact is intuitive for externally tangent circles, but it might be less obvious for internally tangent circles.



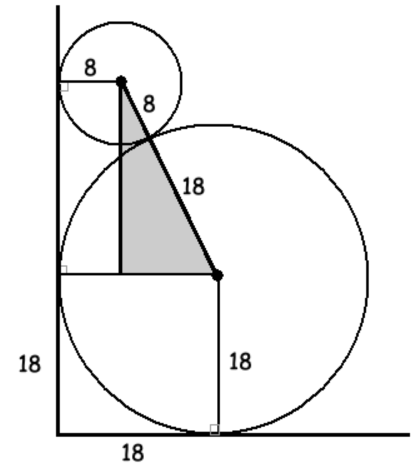
Example 4. A sphere is resting in a corner. A smaller sphere is resting on it as the picture shows. If the radii of the spheres are 7 unit and 17 units, find the vertical distance between the ground and the center of the smaller sphere.



Solution: It is useful to know that any time we intersect a sphere with a plane, we get a circle. We can re-phrase this question in terms of two circles tangents to each other and to a common line.

Let us draw the lines that follow from the tangency. The radius drawn to the vertical line are both horizontal, and the radius drawn to the horizontal line is vertical. Furthermore, if we connect the center of the two circles, the line will contain the point of tangency.

Consider the right triangle shaded on the picture. The length of the hypotenuse is $18+8=26$ units. The length of the horizontal side is $18-8=10$ units. We find the vertical side via the Pythagorean theorem.



$$\begin{aligned} x^2+10^2 &= 26^2 \\ x^2 &= 576 \\ x &= \pm 24 \end{aligned}$$

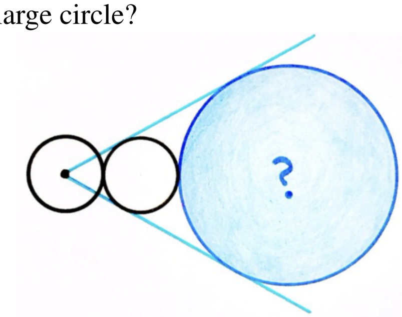
So the vertical side of the shaded triangle is 24 units. The vertical distance is then $24+18=42$ units long.

Practice Problems

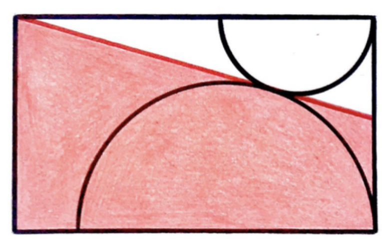
1. Suppose that two circles are externally tangent to each other. Let P be the point of tangency. Let Q be any point on the line tangent to both circles and containing P. Prove that the tangent segments drawn from Q to the two circles are equally long.

The following two puzzles are from the great Catriona Shearer.

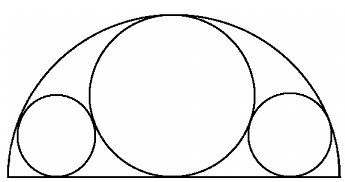
2. The small circles each have area 1. What's the area of the large circle?



3. What fraction of the rectangle is shaded?



4. Three circles in a semicircle. If the semicircle's radius is 12 units, what are the radii of the two congruent smaller circles inscribed?



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