This handout will provide a quick introduction into algebraic expressions.

Definition: A **numerical expression** is an expression that combines numbers and operations.

For example, $3 \cdot 5^2$ is a numerical expression. So are $-\dfrac{12}{3+1}$ and $5^2 - 2^2$ and $-|-5|$. We can **evaluate** numerical expressions by correctly applying the order of operations agreement. It is important that we clearly understand notation.

Example 1: Evaluate each of the given numerical expressions.

a) $3 \cdot 5^2$ b) $-\dfrac{12}{3+1}$ c) $3^2 + 2^2$ d) $(3+2)^2$ e) -3^2 f) $(-3)^2$ g) $-|-5|$

Solution: a) Between exponentiation and multiplication, we first perform the exponentiation. $3 \cdot 5^2 = 3 \cdot 25 = \boxed{75}$

b) The addition in the denominator must be performed before we divide. (Why?) $-\dfrac{12}{3+1} = -\dfrac{12}{4} = \boxed{-6}$

c) $3^2 + 2^2 = 9 + 4 = \boxed{13}$

d) $(3+2)^2 = 5^2 = \boxed{25}$

Note: The error of confusing $3^2 + 2^2$ with $(3+2)^2$ is called the "Freshman's Dream Error".

e) $-3^2 = \boxed{-9}$

f) $(-3)^2 = \boxed{9}$

Note: In looking at -3^2 and $(-3)^2$, we can interpret the minus sign in front of 3 as 'the opposite of'. That is the same as multiplication by -1. Now we can apply order of opertaions, and exponentiation comes before multiplication.

$$-3^2 = -1 \cdot 3^2 = -1 \cdot 9 = -9 \quad \text{but} \quad (-3)^2 = (-3)(-3) = 9$$

In the case of -3^2, we take the opposite of the square of three.
In the case of $(-3)^2$, we square the opposite of three.

g) $-|-5| = \boxed{-5}$ This is a perfect example that two minuses don't always make a plus. What happens here?

Definition: An **algebraic expression** is an expression that combines numbers, operations, and variables.

Variables always represent numbers, so they are subjects to the same rules as numbers. For example, $3x^2 - 1$ is an algebraic expression. So are $-x + 3$ and $2a - b$ and $5y + 3$. We can not automatically evaluate an algebraic expression because we often do not know the value of the variables. For example, the expression $3x^2 - 1$ has different values for different values of x.

Example 2: Evaluate the algebraic expression $3x^2 - 1$ given the values of x.

a) $x = 2$ b) $x = 0$ c) $x = -1$ d) $x = 5$

Solution: a) If $x = 2$, then $3x^2 - 1 = 3 \cdot 2^2 - 1 = 3 \cdot 4 - 1 = 12 - 1 = \boxed{11}$

b) If $x = 0$, then $3x^2 - 1 = 3 \cdot 0^2 - 1 = 3 \cdot 0 - 1 = 0 - 1 = \boxed{-1}$

c) If $x = -1$, then $3x^2 - 1 = 3(-1)^2 - 1 = 3 \cdot 1 - 1 = 3 - 1 = \boxed{2}$

d) If $x = 5$, then $3x^2 - 1 = 3(5)^2 - 1 = 3 \cdot 25 - 1 = 75 - 1 = \boxed{74}$

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Sample Problems

1. (Natural numbers) Evaluate each of the algebraic expressions when $p = 7$ and $q = 3$.

a) $15 - p$ d) $p - 2q$ g) $2q^2$ i) $15 - \dfrac{p+q}{5}$

b) pq e) $p^2 - q^2$

c) $4p - q$ f) $(p-q)^2$ h) $(2q)^2$ j) $(p+q)^2 - (5q-2p)^4$

2. (Integers) Evaluate each of the following numerical expressions.

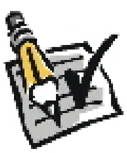
a) $2 - 5(3-7)$ b) $24 - 10 + 2$ c) -4^2 d) $(-4)^2$ e) $|3| - |8|$ f) $|3-8|$

3. Let $a = -4$, $b = 2$, and $x = -3$. Evaluate each of the following expressions.

a) $a^2 - b^2$ b) $(a-b)^2$ c) $a^b - 2bx - x^2 - 2x$ d) $\dfrac{-x^2 + (x+2)^2}{(x-1)}$ e) $\dfrac{x-1}{x+3}$

4. Evaluate each of the following algebrac expressions with the value(s) given.

a) $-x^2 - 5x + 2$ if $x = -2$ b) $-16t^2 + 32t + 240$ if $t = 3$



Practice Problems

1. Evaluate each of the following numerical expressions.

a) $24 - 5 + 1$ b) $24 \div 3 \cdot 2$ c) -1^2 d) $(-1)^2$ e) $-|4| - |7|$ f) $-|4-7|$ g) $6^2 - 4^2$ h) $(6-4)^2$

2. Evaluate each of the algebraic expressions when $x = 6$ and $y = 8$.

a) $19 - y + x$ d) $x^2 + y^2$ g) $3(y-x)$ i) $5x - \dfrac{y}{2}$

b) $19 - (y+x)$ e) $(x+y)^2$

c) $2x^2 - 5y + 3$ f) $3y - x$ h) $\dfrac{5x-y}{2}$ j) $\dfrac{x^2 - 5x + 4}{y-3}$

3. Consider the expression $\dfrac{6x - 3y - xy + 2x^2}{2x - y} - 3$. Evaluate this expression if

a) $x = 5$ and $y = 1$ b) $x = 5$ and $y = 2$ c) $x = 5$ and $y = 3$ d) $x = 4$ and $y = 1$

4. Consider the expression $\dfrac{6x - 3y - xy + 2x^2}{2x - y} - 3$. Evaluate this expression if

a) $x = -1$ and $y = 2$ b) $x = -3$ and $y = -6$ c) $x = 3$ and $y = -2$ d) $x = -7$ and $y = 4$

5. Evaluate $-m^2 - m$ if

a) $m = 2$ b) $m = -2$ c) $m = 0$ d) $m = 5$ e) $m = -5$

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6. Evaluate $\dfrac{8x+x^2-33}{x+11}$ if

a) $x=0$ b) $x=7$ c) $x=-4$ d) $x=-11$ e) $x=-1$

7. a) It is a common mistake to think that the expressions $2x-3$ and $2x+3$ are opposites. They are not. Evaluate these expressions for the values given below to fill out the table below.

	$x=2$	$x=5$	$x=6$	$x=10$	$x=-1$	$x=-5$	$x=-8$
$2x-3$	1						
$2x+3$	7						

b) the opposite of $2x-3$ is actually $-2x+3$. Evaluate these expressions for the values given below to fill out the table below.

	$x=2$	$x=5$	$x=6$	$x=10$	$x=-1$	$x=-5$	$x=-8$
$2x-3$							
$-2x+3$							

8. Evaluate $\dfrac{x-2}{2-x}$ if

a) $x=0$ b) $x=10$ c) $x=2$ d) $x=-13$

9. Evaluate each of the following algebrac expressions with the value(s) given.

a) $3x^2-x+7$ if $x=-1$ b) $-a+5b$ if $a=3$ and $b=-2$ c) $\dfrac{x^x-1}{x-1}$ if $x=2$

Answers

Sample Problems

1\. 8 b) 21 c) 25 d) 1 e) 40 f) 16 g) 18 h) 36 i) 13 j) 99

2\. a) 22 b) 16 c) -16 d) 16 e) -5 f) 5 3. a) 12 b) 36 c) 25 d) 2 e) undefined

4\. a) 8 b) 192

Practice Problems

1\. a) 20 b) 16 c) -1 d) 1 e) -11 f) -3 g) 20 h) 4 i) 13 j) 17

2\. a) 17 b) 5 c) 35 d) 100 e) 196 f) 18 g) 6 h) 11 i) 26 j) 2

3\. a) 5 b) 5 c) 5 d) 4 4. a) -1 b) undefined c) 3 d) -7

5\. a) -6 b) -2 c) 0 d) -30 e) -20 6. a) -3 b) 4 c) -7 d) undefined e) -4

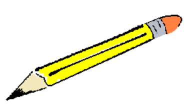
7\. a)

	$x=2$	$x=5$	$x=6$	$x=10$	$x=-1$	$x=-5$	$x=-8$
$2x-3$	1	7	9	17	-5	-13	13
$2x+3$	7	13	15	23	1	-7	-19

b)

	$x=2$	$x=5$	$x=6$	$x=10$	$x=-1$	$x=-5$	$x=-8$
$2x-3$	1	7	9	17	-5	-13	13
$-2x+3$	-1	-7	-9	-17	5	13	-13

8\. a) -1 b) -1 c) undefined d) -1 9. a) 11 b) -13 c) 3



Solutions - Sample Problems

1. Evaluate each of the algebraic expressions when $p=7$ and $q=3$.

   a) $15-p$
   Solution: Step 1. We re-write the expression with one modification: we replace each variable by an empty pair of parenthses.
   Step 2. We insert the values into the parentheses. Now the problem becomes an order of operations problem.
   Step 3. We drop the unnecessary parentheses and work out the order of operations problem. (It may appear awkward to create these parentheses but they will later become extremely helpful.)

$$\begin{array}{lrcl} \text{Step 1.} & 15-p & = & 15-(\ ) \\ \text{Step 2.} & & = & 15-(7) \\ \text{Step 3.} & & = & 15-7 \\ & & = & \boxed{8} \end{array}$$

b) pq
Solution:

$$\begin{aligned} &\text{Step 1.} & pq &= (\ )(\ ) \\ &\text{Step 2.} & &= (7)(3) \\ &\text{Step 3.} & &= \boxed{21} \end{aligned}$$

c) $4p-q$
Solution:

$$\begin{aligned} &\text{Step 1.} & 4p-q &= 4(\ )-(\ ) & \\ &\text{Step 2.} & &= 4(7)-(3) & \\ &\text{Step 3.} & &= 4\cdot 7-3 & \text{multiplication} \\ & & &= 28-3 & \text{subtraction} \\ & & &= \boxed{25} & \end{aligned}$$

d) $p-2q$
Solution:

$$\begin{aligned} &\text{Step 1.} & p-2q &= (\ )-2(\ ) & \\ &\text{Step 2.} & &= (7)-2(3) & \\ &\text{Step 3.} & &= 7-2\cdot 3 & \text{multiplication} \\ & & &= 7-6 & \text{subtraction} \\ & & &= \boxed{1} & \end{aligned}$$

e) p^2-q^2
Solution:

$$\begin{aligned} p^2-q^2 &= (\ )^2-(\ )^2 & \\ &= (7)^2-(3)^2 & \\ &= 7^2-3^2 & \text{exponents,} \\ &= 49-3^2 & \text{left to right} \\ &= 49-9 & \text{subtraction} \\ &= \boxed{40} & \end{aligned}$$

f) $(p-q)^2$
Solution:

$$\begin{aligned} (p-q)^2 &= [(\ )-(\ )]^2 & \\ &= [(7)-(3)]^2 & \\ &= (7-3)^2 & \text{subtraction in parentheses} \\ &= 4^2 & \text{exponentiation} \\ &= \boxed{16} & \end{aligned}$$

g) $2q^2$
Solution:

$$\begin{aligned} 2q^2 &= 2(\ )^2 & \\ &= 2(3)^2 & \\ &= 2\cdot 3^2 & \text{exponentiation} \\ &= 2\cdot 9 & \text{multiplication} \\ &= \boxed{18} & \end{aligned}$$

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h) $(2q)^2$
Solution:

$$\begin{aligned}(2q)^2 &= [2(\ )]^2 \\ &= [2(3)]^2 \\ &= (2\cdot 3)^2 && \text{multiplication in parentheses} \\ &= 6^2 && \text{exponents} \\ &= \boxed{36}\end{aligned}$$

i) $15-\dfrac{p+q}{5}$
Solution: From here on, we show computations **in the form they should appear**. Once you wrote down the expression with little parentheses instead of the letters, you can insert the values into it.

$$\begin{aligned}15-\frac{p+q}{5} &= 15-\frac{(\ )+(\ )}{5} \\ &= 15-\frac{(7)+(3)}{5} \\ &= 15-\frac{7+3}{5} && \text{invisible parentheses!} \\ &= 15-\frac{10}{5} && \text{division} \\ &= 15-2 && \text{subtraction} \\ &= \boxed{13}\end{aligned}$$

j) $(p+q)^2-(5q-2p)^4$
Solution:

$$\begin{aligned}(p+q)^2-(5q-2p)^4 &= [(7)+(3)]^2-[5(3)-2(7)]^4 \\ &= (7+3)^2-(5\cdot 3-2\cdot 7)^4 && \text{addition in parentheses} \\ &= 10^2-(5\cdot 3-2\cdot 7)^4 && \text{multiplications in parentheses} \\ &= 10^2-(15-2\cdot 7)^4 && \text{left to right}\end{aligned}$$

$$\begin{aligned}&= 10^2-(15-14)^4 && \text{subtraction in parentheses} \\ &= 10^2-1^4 && \text{exponents, left to right} \\ &= 100-1^4 && \text{careful! } 1^4\neq 4 \\ &= 100-1 \\ &= \boxed{99}\end{aligned}$$

2. (Integers) Evaluate each of the following numerical expressions.

a) $2-5(3-7)$

Solution: We will apply order of operations. First we perform the subtraction in the parentheses.

$$\begin{aligned}2-5(3-7) &= && \text{subtraction in parentheses} \\ 2-5(-4) &= && \text{multiplication} \\ 2-(-20) &= && \text{subtraction} \\ 2+20 &= \boxed{22}\end{aligned}$$

b) $24 - 10 + 2$

Solution: It is NOT true that addition comes before subtraction. Addition and subtraction are equally strong, so between those two, we perform them left to right. First come, first served.

$$24 - 10 + 2 = 14 + 2 = \boxed{16}$$

c) -4^2

Solution: as it was discussed before, -4^2 is quite different from $(-4)^2$. This is $-1 \cdot 4^2 = \boxed{-16}$.

d) $(-4)^2$

This is when -4 is squared. So $(-4)^2 = -4(-4) = \boxed{16}$

e) $|3| - |8|$

Solution: We subtract the absolute value of 8 from the absolute value of 3. So $|3| - |8| = 3 - 8 = \boxed{-5}$

f) $|3 - 8|$

Solution: This is the absolute value of the difference. Absolute value signs also function of grouping symbols (i.e. parentheses) to overwrite the usual order of operations. So $|3 - 8| = |-5| = \boxed{5}$

3. Let $a = -4$, $b = 2$, and $x = -3$. Evaluate each of the following expressions.

a) $a^2 - b^2$

Solution: First we re-write the expression with one change, we write little pairs of parentheses instead of the letters.

$$a^2 - b^2 = (\ \ )^2 - (\ \ )^2$$

We now write the values inside the parentheses. From here on this is an order of operations problem.

$$\begin{aligned} a^2 - b^2 &= (-4)^2 - (2)^2 && \text{drop extra parentheses} \\ &= (-4)^2 - 2^2 && \text{exponents} \\ &= 16 - 4 && \text{subtraction} \\ &= \boxed{12} \end{aligned}$$

b) $(a - b)^2$

Solution: First we re-write the expression with one change, we write little pairs of parentheses instead of the letters.

$$(a - b)^2 = ((\ \ ) - (\ \ ))^2$$

We now write the values inside the parentheses. From here on this is an order of operations problem.

$$\begin{aligned} (a - b)^2 &= ((-4) - (2))^2 && \text{drop extra parentheses} \\ &= (-4 - 2)^2 && \text{subtraction in parentheses} \\ &= (-6)^2 && \text{exponent} \\ &= \boxed{36} \end{aligned}$$

This and the previous problem is here to remind you that $(a - b)^2$ and $a^2 - b^2$ are two different expressions.

c) $a^b - 2bx - x^2 - 2x$

Solution: First we re-write the expression with one modification only: we write little pairs of parentheses instead of the letters.

$$a^b - 2bx - x^2 - 2x = (\ \ )^{(\ \ )} - 2(\ \ )(\ \ ) - (\ \ )^2 - 2(\ \ )$$

We now write the values inside the parentheses. From here on this is an order of operations problem.
$a^b - 2bx - x^2 - 2x =$

$$\begin{aligned}
&= (\ )^{(\ )} - 2(\ )(\ ) - (\ )^2 - 2(\ ) \\
&= (-4)^{(2)} - 2(2)(-3) - (-3)^2 - 2(-3) && \text{drop extra parentheses} \\
&= (-4)^2 - 2\cdot 2(-3) - (-3)^2 - 2(-3) && \text{exponents, left to right} \\
&= 16 - 2\cdot 2(-3) - (-3)^2 - 2(-3) \\
&= 16 - 2\cdot 2(-3) - 9 - 2(-3) && \text{multiplications, left to right} \\
&= 16 - 4(-3) - 9 - 2(-3) \\
&= 16 - (-12) - 9 - 2(-3)
\end{aligned}$$

$$\begin{aligned}
&= 16 - (-12) - 9 - (-6) && \text{additions, subtractions, left to right} \\
&= 16 + 12 - 9 - (-6) \\
&= 28 - 9 - (-6) \\
&= 19 - (-6) \\
&= 19 + 6 \\
&= \boxed{25}
\end{aligned}$$

d) $\dfrac{-x^2 + (x+2)^2}{(x-1)}$

Solution: First we re-write the expression with only one modification: we write little pairs of parentheses instead of the letters.

$$\frac{-x^2 + (x+2)^2}{(x-1)} = \frac{-(\ )^2 + ((\ )+2)^2}{((\ )-1)}$$

We now write the values inside the parentheses. From here on this is an order of operations problem.

$$\begin{aligned}
\frac{-x^2 + (x+2)^2}{(x-1)} &= \frac{-(-3)^2 + ((-3)+2)^2}{((-3)-1)} && \text{drop parentheses} \\
&= \frac{-(-3)^2 + (-3+2)^2}{(-3-1)} && \text{addition in parentheses upstairs} \\
&= \frac{-(-3)^2 + (-1)^2}{(-3-1)} && \text{subtraction downstairs in parentheses} \\
&= \frac{-(-3)^2 + (-1)^2}{(-4)} && \text{drop parentheses} \\
&= \frac{-(-3)^2 + (-1)^2}{-4} && \text{exponents upstairs} \\
&= \frac{-9+1}{-4} && \text{addition} \\
&= \frac{-8}{-4} && \text{division} \\
&= \boxed{2}
\end{aligned}$$

e) $\dfrac{x-1}{x+3}$

Solution: First we re-write the expression with only one modification: we write little pairs of parentheses instead of the letters.

$$\frac{x-1}{x+3}=\frac{(\ )-1}{(\ )+3}$$

We write the values inside the parentheses and evaluate the expression.

$$\frac{x-1}{x+3}=\frac{(-3)-1}{(-3)+3}=\frac{-4}{0}=\boxed{\text{undefined}}$$

4. Evaluate each of the following algebrac expressions with the value(s) given.

a) $-x^2-5x+2$ if $x=-2$

Solution: We substitute -2 into the expression. Please note that if the value of x is negative, we will need to place parentheses around it.

$$-x^2-5x+2=-(-2)^2-5(-2)+2$$

According to order of operations, we perform the exponentiation first.

$$-(-2)^2-5(-2)+2=-4-5(-2)+2=-4-(-10)+2=-4+10+2=6+2=\boxed{8}$$

Why don't the two minuses make a plus in $-(-2)^2$? They do, it's just that there are three minus signs and not two: $-(-2)^2=-1(-2)(-2)$.

b) $-16t^2+32t+240$ if $t=3$

Solution: We substitute 3 into the expression. Please note that if the value of x is negative, we will need to place parentheses around it.

$$\begin{aligned}-16t^2+32t+240 &= -16(3)^2+32(3)+240\\ &= -16\cdot 9+32\cdot 3+240\\ &= -144+96+240=-48+240=\boxed{192}\end{aligned}$$

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