# Part 1 - The History Thus Far and the Problem

Recall what we know about exponentiation thus far. Exponential notation expresses repeated multiplication.

**Definition**: We define $2^7$ to denote the factor 2 multiplied by itself repeatedly, such as

$$\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{\text{7 factors}} = 2^7$$

When mathematicians agreed to this definition, that was a free choice. They could have gone with other definitions. Once this definition exists, however, certain properties are automatically true, and we have no other option but to recognize them as true. They just fell into our laps.

**Theorem 1.** If $a$ is any number and $m$, $n$ are any positive integers, then $a^n \cdot a^m = a^{n+m}$

**Theorem 2.** If $a$ is any non-zero number and $m$, $n$ are any positive integers, then $\dfrac{a^n}{a^m} = a^{n-m}$

**Theorem 3.** If $a$ is any number and $m$, $n$ are any positive integers, then $(a^n)^m = a^{nm}$

**Theorem 4.** If $a$, $b$ are any numbers and $n$ is any positive integer, then $(ab)^n = a^n b^n$

**Theorem 5.** If $a$, $b$ are any numbers, $b \neq 0$, and $n$ is any positive integer, then $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Again, the definition, immediately followed by the theorems. And then there was a quiet. Another opening for a free choice.

Consider the expression $2^x$. The problem is that the definition of exponentiation only allows for a positive integer value of $x$. The expression $2^x$ is meaningful for $x = 2$ or 9 or 100, but it is not meaningful for values of $x$ such as $-3$ or $\dfrac{3}{5}$ or $3.2$. In short, the world of exponents was just the set of all natural numbers. Mathematicians usually don't like that. The best case scenario, the ultimate hope is that the definition of exponents could be extended to any number for $x$. That way, $2^x$ would be meaningful, no matter what the value of $x$ is.

So, one of the issues was the desire to grow our world of exponents beyond the set of all natural numbers. This will be achieved in several steps. Today, we are only focusing on enlarging the world of exponents from $\mathbb{N}$ to $\mathbb{Z}$ (i.e. from the set of all natural numbers to the set of all integers).

The other issue was that as we enlarge our world, we pay especial attention that the new definitions will not conflict with the mathematics we already have. This principle comes up often in our choices, and it is sometimes called the **expansion principle**.

**Definition**: In many situations, mathematicians attempt to increase, to enlarge our world. The **expansion principle** is that when we enlarge our mahematics by adding new definitions, we do so in such a way that the new definitions never create conflicts with the mathematics we already have.

# Part 2 - Integer Exponents

Suppose we want to define $2^0$. The repeated multiplication definition can not be applied to zero, so we have complete freedom to define $2^0$. As it turns out, if we insist on a definition that does not conflict with Rule 2, $\dfrac{a^n}{a^m} = a^{n-m}$, then we do not have all that many choices for $2^0$. Let us think of zero as the result of the subtraction $3-3$, and that we would like to define $2^0$ so that Rule 2 is still true.

$$2^0 = 2^{3-3} \overset{\text{rule 2}}{=} \frac{2^3}{2^3} = \frac{8}{8} = 1$$

This is an expansion principle proof. It did not prove that the value of $2^0$ is or must be zero. It showed much less; that if we wanted to define $2^0$ without harming Rule 2 in the example given, then the only possible value for $2^0$ is 1. The reader should imagine a team of mathematicians making first sure that no part of our good old math is hurt if we define $2^0 = 1$. And as it turned out, this is exactly the case.

This computation can be repeated with many different bases. For example,

$$5^0 = 5^{2-2} \overset{\text{rule 2}}{=} \frac{5^2}{5^2} = \frac{25}{25} = 1 \quad \text{or} \quad (-3)^0 = (-3)^{2-2} \overset{\text{rule 2}}{=} \frac{(-3)^2}{(-3)^2} = \frac{9}{9} = 1$$

The only base that is problematic is 0. Indeed, division by zero is not allowed and Rule 2, $\dfrac{a^n}{a^m} = a^{n-m}$ does not work with $a = 0$. If we try to perform the same computation with zero, we ultimately end up in $\dfrac{0}{0}$ which is undefined.

**Theorem 6.** If $a$ is any non-zero number, then $a^0 = 1$.

$0^0$ is undefined.

Please note that as we extend our world of exponents, old issues might re-surface. For example, $(-3)^0 = 1$ but $-3^0 = -1$ is an important distinction, but not a new one.

Now that we have defined zero exponent, we will similarly try to define negative integer exponents such as $2^{-3}$.

Again, the original definition can not be applied. We cannot write down the factor two negative three times. So we have a freedom here to define $2^{-3}$ in any way we wish. In this decision, we will again use the expansion principle: that we would like to keep our old rules after having $2^{-3}$ defined.

We will again use Rule 2, $\dfrac{a^n}{a^m} = a^{n-m}$ and write $-3$ as a subtraction between two positive integers.

$$2^{-3} = 2^{1-4} \overset{\text{Rule 2}}{=} \frac{2^1}{2^4} = \frac{2}{16} = \frac{1}{8} = \frac{1}{2^3} \quad \text{or, more elegantly, } 2^{-3} = 2^{1-4} \overset{\text{Rule 2}}{=} \frac{2^1}{2^4} = \frac{\cancel{2}}{\cancel{2} \cdot 2 \cdot 2 \cdot 2} = \frac{1}{2^3}$$

When we discovered this rule, we saw that it was true because of cancellation. In case of a negative exponent, we have the same cancellation, it's just that we run out of factors in the numerator first. The computation can be repated with any base except for zero.

**Theorem 7.** If $a$ is any non-zero number, and $n$ is any positive integer, then $a^{-n} = \dfrac{1}{a^n}$.

$0^{-n}$ is undefined.

## Sample Problems

Simplify each of the following. Assume that all variables represent positive numbers. Present your answer without negative exponents.

1. $3^{-2}$
2. $\dfrac{1}{2^{-3}}$
3. $m^{-4}$
4. $\dfrac{1}{x^{-5}}$
5. $a^8 \cdot a^{-1}$
6. $p^3\left(p^{-7}\right)p^8$
7. $\dfrac{x^{-4}}{x^{-9}}$
8. $\dfrac{50a^{12}}{10a^{-3}}$
9. $\dfrac{t^{-3}}{t^4}$
10. $x^0$
11. $-x^0$
12. $(-x)^0$
13. $\left(b^{-5}\right)\left(b^2\right)\left(b^{-1}\right)$
14. $\dfrac{1}{\left(b^{-5}\right)\left(b^2\right)\left(b^{-1}\right)}$
15. $\dfrac{m^{-2}}{m^{-5}}$
16. $\dfrac{x^3y^{-5}}{z^{-4}}$
17. $\dfrac{18q^3}{6q^{-3}}$
18. $\left(\dfrac{2}{3}\right)^{-3}$
19. $2y^{-3}$
20. $(2y)^{-3}$
21. $\left(-\dfrac{3}{5}\right)^{-2}$
22. $\dfrac{a^3b^{-5}}{a^{-2}b^3}$
23. $\left(3m^3\right)^{-2}$
24. $\left(-2ab^{-3}\right)^{-3}$
25. $\dfrac{\left(k^3\right)^{-3}}{\left(k^{-5}\right)^2}$
26. $\left(\dfrac{2a^{-3}b^5}{-3a^3b^{-2}}\right)^{-2}\left(a^3b^{-5}\right)^{-3}$
27. $\left(-2a^{-3}\right)\left(-2a^{-2}b\right)^{-4}$
28. $\dfrac{\left(-3p^3q^5\right)^2}{\left(2q^0p^3\right)^{-1}}$
29. $\left(\dfrac{2a^{-2}b^3}{-2^2\left(a^{-1}b\right)^{-3}}\right)^{-2}$
30. $\left(-\dfrac{x^3y^0x^{-5}}{y^{-3}}\right)^{-2}$
31. $\left(-\dfrac{x^3y^7x^{-5}}{y^{-3}}\right)^{0}$
32. $\dfrac{x^{-1}+y^{-1}}{x^{-2}-y^{-2}}$
33. $\dfrac{\left(-2a^{-2}\right)^{-2}b^3a^0\left(-aba^{-2}b^{-2}\right)^{-3}}{2a^2\left(-2a^{-2}b\right)^{-2}ab^0}$
34. $\left(\dfrac{-a^2\left(b^{-1}a\right)^{-5}}{b^7\left(-ab^2\right)^{-3}}\right)^{-2}$
35. $\dfrac{\left(x^{-2}\right)^{-2}y^3x^0\left(-2yx^0y^{-2}x^{-2}\right)^0}{yx^5\left(y^{-2}x\right)^{-3}\left(2x^{-1}yx^3\right)^{-1}}$

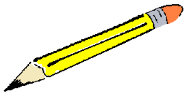
 

# Answers

## Discussion

## Sample Problems

1. $\frac{1}{9}$
2. $8$
3. $\frac{1}{m^4}$
4. $x^5$
5. $a^7$
6. $p^4$
7. $x^5$
8. $5a^{15}$
9. $\frac{1}{t^7}$
10. $1$
11. $-1$
12. $1$
13. $\frac{1}{b^4}$
14. $b^4$
15. $m^3$
16. $\frac{x^3z^4}{y^5}$
17. $3q^6$
18. $\frac{27}{8}$
19. $\frac{2}{y^3}$
20. $\frac{1}{8y^3}$
21. $\frac{25}{9}$
22. $\frac{a^5}{b^8}$
23. $\frac{1}{9m^6}$
24. $-\frac{b^9}{8a^3}$
25. $k$
26. $\frac{9}{4}a^3b$
27. $-\frac{a^5}{8b^4}$
28. $18p^9q^{10}$
29. $\frac{4a^{10}}{b^{12}}$
30. $\frac{x^4}{y^6}$
31. $1$
32. $\frac{xy}{y-x}$
33. $-\frac{b^8}{2}$
34. $\frac{1}{b^8}$
35. $\frac{2x^4}{y^3}$



# Sample Problems - Solutions

Simplify each of the following. Assume that all variables represent positive numbers. Present your answer without negative exponents.

1. $3^{-2}$

   Solution: We just apply the rule $a^{-n} = \dfrac{1}{a^n}$.

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

2. $\dfrac{1}{2^{-3}}$

   Solution: We apply the rule $a^{-n} = \dfrac{1}{a^n}$.

$$\frac{1}{2^{-3}} = \frac{1}{\dfrac{1}{2^3}} = \frac{1}{\dfrac{1}{8}}$$

   To divide is to multiply by the reciprocal:

$$\frac{1}{\dfrac{1}{8}} = 1 \cdot \frac{8}{1} = 8$$

   This is true in general: $\dfrac{1}{a^{-n}} = a^n$

$$\frac{1}{a^{-n}} = \frac{1}{\dfrac{1}{a^n}} = 1 \cdot \frac{a^n}{1} = a^n$$

3. $m^{-4}$

   Solution: We apply the rule $a^{-n} = \dfrac{1}{a^n}$.

$$m^{-4} = \frac{1}{m^4}$$

4. $\dfrac{1}{x^{-5}}$

   Solution: We have already proven that $\dfrac{1}{a^{-n}} = a^n$

$$\frac{1}{x^{-5}} = x^5$$

5. $a^8 \cdot a^{-1}$

   Solution 1: We can apply the rule $a^n \cdot a^m = a^{n+m}$

$$a^8 \cdot a^{-1} = a^{8+(-1)} = a^7$$

   Solution 2: We can apply the rule $a^{-n} = \dfrac{1}{a^n}$ and then the rule $\dfrac{a^n}{a^m} = a^{n-m}$.

$$a^8 \cdot a^{-1} = a^8 \cdot \frac{1}{a^1} = \frac{a^8}{1} \cdot \frac{1}{a} = \frac{a^8}{a} = \frac{a^8}{a^1} = a^{8-1} = a^7$$

6. $p^3\left(p^{-7}\right)p^8$

Solution 1: We can apply the rule $a^n \cdot a^m = a^{n+m}$

$$p^3\left(p^{-7}\right)p^8 = p^{3+(-7)+8} = p^4$$

Solution 2: We can apply the rules $a^{-n} = \dfrac{1}{a^n}$ and $a^n \cdot a^m = a^{n+m}$ and $\dfrac{a^n}{a^m} = a^{n-m}$.

$$p^3\left(p^{-7}\right)p^8 = p^3 \cdot \frac{1}{p^7} \cdot p^8 = \frac{p^3}{1} \cdot \frac{1}{p^7} \cdot \frac{p^8}{1} = \frac{p^3 \cdot p^8}{p^7} = \frac{p^{3+8}}{p^7} = \frac{p^{11}}{p^7} = p^{11-7} = p^4$$

7. $\dfrac{x^{-4}}{x^{-9}}$

Solution 1: We can apply the rule $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{x^{-4}}{x^{-9}} = x^{-4-(-9)} = x^{-4+9} = x^5$$

Solution 2: We can apply the rules $a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{x^{-4}}{x^{-9}} = \frac{x^9}{x^4} = x^{9-4} = x^5$$

8. $\dfrac{50a^{12}}{10a^{-3}}$

Solution 1: We can apply the rule $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{50a^{12}}{10a^{-3}} = 5a^{12-(-3)} = 5a^{12+3} = 5a^{15}$$

Solution 2: We can apply the rules $a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{50a^{12}}{10a^{-3}} = \frac{50a^{12}a^3}{10} = 5a^{12+3} = 5a^{15}$$

9. $\dfrac{t^{-3}}{t^4}$

Solution 1: We can apply the rules $\dfrac{a^n}{a^m} = a^{n-m}$ and then $a^{-n} = \dfrac{1}{a^n}$.

$$\frac{t^{-3}}{t^4} = t^{-3-4} = t^{-7} = \frac{1}{t^7}$$

Solution 2: We can apply the rule $a^{-n} = \dfrac{1}{a^n}$ and then $a^n \cdot a^m = a^{n+m}$.

$$\frac{t^{-3}}{t^4} = \frac{1}{t^4 \cdot t^3} = \frac{1}{t^7}$$

10. $x^0$

Solution: There is a separate rule stating that as long as $x$ is not zero, then $x^0 = 1$. So the answer is $1$.

11. $-x^0$

Solution: This is the opposite of $x^0$ and so the answer is $-1$.

$$-x^0 = -1 \cdot x^0 = -1 \cdot 1 = -1$$

12. $(-x)^0$

Solution: This is again $1$ because any non-zero riased to the power zero is $1$.

13. $\left(b^{-5}\right)\left(b^{2}\right)\left(b^{-1}\right)$

Solution 1: We can apply the rules $a^n \cdot a^m = a^{n+m}$ and then $a^{-n} = \dfrac{1}{a^n}$.

$$\left(b^{-5}\right)\left(b^{2}\right)\left(b^{-1}\right) = b^{-5+2+(-1)} = b^{-4} = \frac{1}{b^4}$$

Solution 2: We can apply the rule $a^{-n} = \dfrac{1}{a^n}$ and then just cancel.

$$\left(b^{-5}\right)\left(b^{2}\right)\left(b^{-1}\right) = \frac{1}{b^5} \cdot b^2 \cdot \frac{1}{b^1} = \frac{1}{b^5} \cdot \frac{b^2}{1} \cdot \frac{1}{b^1} = \frac{b^2}{b^6} = \frac{\cancel{b} \cdot \cancel{b}}{\cancel{b} \cdot \cancel{b} \cdot b \cdot b \cdot b \cdot b} = \frac{1}{b^4}$$

14. $\dfrac{1}{\left(b^{-5}\right)\left(b^{2}\right)\left(b^{-1}\right)}$

Solution 1: We can apply the rules $a^n \cdot a^m = a^{n+m}$ and then $a^{-n} = \dfrac{1}{a^n}$.

$$\frac{1}{\left(b^{-5}\right)\left(b^{2}\right)\left(b^{-1}\right)} = \frac{1}{b^{-5+2+(-1)}} = \frac{1}{b^{-4}} = \frac{1}{\dfrac{1}{b^4}} = 1 \cdot \frac{b^4}{1} = b^4$$

Solution 2: We can apply the rule $a^{-n} = \dfrac{1}{a^n}$ and then $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{1}{\left(b^{-5}\right)\left(b^{2}\right)\left(b^{-1}\right)} = \frac{b^5 \cdot b^1}{b^2} = \frac{b^6}{b^2} = b^{6-2} = b^4$$

15. $\dfrac{m^{-2}}{m^{-5}}$

Solution 1: We can apply the rules $\dfrac{a^n}{a^m} = a^{n-m}$ and then $a^{-n} = \dfrac{1}{a^n}$.

$$\frac{m^{-2}}{m^{-5}} = m^{-2-(-5)} = m^{-2+5} = m^3$$

Solution 2: We can apply the rule $a^{-n} = \dfrac{1}{a^n}$ and then $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{m^{-2}}{m^{-5}} = \frac{m^5}{m^2} = m^{5-2} = m^3$$

16. $\dfrac{x^3y^{-5}}{z^{-4}}$

Solution: Each variable occurs only once and so this problem is just about bringing it to the form required. We can apply the rule $a^{-n}=\dfrac{1}{a^n}$. We hve alread shown that $\dfrac{1}{a^{-n}}=a^n$.

$$\frac{x^3y^{-5}}{z^{-4}}=\frac{x^3z^4}{y^5}$$

17. $\dfrac{18q^3}{6q^{-3}}$

Solution 1: We can apply the rule $\dfrac{a^n}{a^m}=a^{n-m}$.

$$\frac{18q^3}{6q^{-3}}=\frac{\cancel{6}\cdot 3q^{3-(-3)}}{\cancel{6}\cdot 1}=\frac{3q^{3+3}}{1}=3q^6$$

Solution 2: We can apply the rules $a^{-n}=\dfrac{1}{a^n}$ and then $a^n\cdot a^m=a^{n+m}$.

$$\frac{18q^3}{6q^{-3}}=\frac{\cancel{6}\cdot 3q^3q^3}{\cancel{6}\cdot 1}=3q^6$$

18. $\left(\dfrac{2}{3}\right)^{-3}$

Solution: We can apply the rule $a^{-n}=\dfrac{1}{a^n}$.

$$\left(\frac{2}{3}\right)^{-3}=\frac{1}{\left(\frac{2}{3}\right)^3}=\frac{1}{\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}}=\frac{1}{\frac{8}{27}}=1\cdot\frac{27}{8}=\frac{27}{8}$$

Note that we basically proved here that $\left(\dfrac{a}{b}\right)^{-n}=\left(\dfrac{b}{a}\right)^n$.

19. $2y^{-3}$

Solution: We can apply the rule $a^{-n}=\dfrac{1}{a^n}$. It is important to note that the base of exponentiation is $y$ and not $2y$.

$$2y^{-3}=2\cdot\frac{1}{y^3}=\frac{2}{1}\cdot\frac{1}{y^3}=\frac{2}{y^3}$$

20. $(2y)^{-3}$

Solution: We can apply the rule $a^{-n}=\dfrac{1}{a^n}$. This time the base of exponentiation is $2y$. So we will apply the rule $(ab)^n=a^nb^n$.

$$(2y)^{-3}=\frac{1}{(2y)^3}=\frac{1}{2^3y^3}=\frac{1}{8y^3}$$

21. $\left(-\dfrac{3}{5}\right)^{-2}$

Solution 1: We can apply the rule $a^{-n}=\dfrac{1}{a^n}$.

$$\left(-\frac{3}{5}\right)^{-2}=\frac{1}{\left(-\frac{3}{5}\right)^{2}}=\frac{1}{\left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right)}=\frac{1}{\frac{-3}{5}\cdot\frac{-3}{5}}=\frac{1}{\frac{9}{25}}=1\cdot\frac{25}{9}=\frac{25}{9}$$

Solution 2: We proved previously that $\left(\dfrac{a}{b}\right)^{-n}=\left(\dfrac{b}{a}\right)^{n}$. Using that,

$$\left(-\frac{3}{5}\right)^{-2}=\left(-\frac{5}{3}\right)^{2}=\left(-\frac{5}{3}\right)\left(-\frac{5}{3}\right)=\frac{25}{9}$$

22. $\dfrac{a^3b^{-5}}{a^{-2}b^3}$

Solution 1: We can apply the rule $\dfrac{a^n}{a^m}=a^{n-m}$ and then $a^{-n}=\dfrac{1}{a^n}$.

$$\frac{a^3b^{-5}}{a^{-2}b^3}=a^{3-(-2)}b^{-5-3}=a^{3+2}b^{-5-3}=a^5b^{-8}=a^5\cdot\frac{1}{b^8}=\frac{a^5}{1}\cdot\frac{1}{b^8}=\frac{a^5}{b^8}$$

Solution 2: We can apply the rules $a^{-n}=\dfrac{1}{a^n}$ and $a^n\cdot a^m=a^{n+m}$.

$$\frac{a^3b^{-5}}{a^{-2}b^3}=\frac{a^3a^2}{b^3b^5}=\frac{a^5}{b^8}$$

23. $\left(3m^3\right)^{-2}$

Solution: We can apply the rule $a^{-n}=\dfrac{1}{a^n}$ and then $(ab)^n=a^nb^n$ and also $(a^n)^m=a^{nm}$.

$$\left(3m^3\right)^{-2}=\frac{1}{\left(3m^3\right)^2}=\frac{1}{3^2\left(m^3\right)^2}=\frac{1}{9m^{3\cdot 2}}=\frac{1}{9m^6}$$

24. $\left(-2ab^{-3}\right)^{-3}$

Solution: We can apply the rule $(ab)^n=a^nb^n$ and then $(a^n)^m=a^{nm}$.

$$\left(-2ab^{-3}\right)^{-3}=(-2)^{-3}a^{-3}\left(b^{-3}\right)^{-3}=(-2)^{-3}a^{-3}b^{-3(-3)}=(-2)^{-3}a^{-3}b^{9}$$

We now apply $a^{-n}=\dfrac{1}{a^n}$.

$$(-2)^{-3}a^{-3}b^9=\frac{1}{(-2)^3}\cdot\frac{1}{a^3}\cdot b^9=\frac{1}{-8}\cdot\frac{1}{a^3}\cdot\frac{b^9}{1}=\frac{b^9}{-8a^3}=-\frac{b^9}{8a^3}$$

25. $\dfrac{\left(k^3\right)^{-3}}{\left(k^{-5}\right)^2}$

Solution: We can apply the rule $(a^n)^m=a^{nm}$ and then $\dfrac{a^n}{a^m}=a^{n-m}$.

$$\frac{\left(k^3\right)^{-3}}{\left(k^{-5}\right)^2}=\frac{k^{3(-3)}}{k^{-5\cdot 2}}=\frac{k^{-9}}{k^{-10}}=k^{-9-(-10)}=k^{-9+10}=k^1=k$$

26. $\left(\dfrac{2a^{-3}b^5}{-3a^3b^{-2}}\right)^{-2}\left(a^3b^{-5}\right)^{-3}$

Solution:

$$\begin{aligned}
E &= \left(\frac{2a^{-3}b^5}{-3a^3b^{-2}}\right)^{-2}\left(a^3b^{-5}\right)^{-3} = \left(\frac{2a^{-3-3}b^{5-(-2)}}{-3}\right)^{-2}\left(a^3b^{-5}\right)^{-3} && \text{apply } \frac{a^n}{a^m}=a^{n-m}\\
&= \left(\frac{2a^{-6}b^{5+2}}{-3}\right)^{-2}\left(a^3b^{-5}\right)^{-3}\\
&= \left(\frac{2a^{-6}b^{7}}{-3}\right)^{-2}\left(a^3b^{-5}\right)^{-3} && \text{apply } \left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}\\
&= \left(\frac{-3}{2a^{-6}b^{7}}\right)^{2}\left(a^3b^{-5}\right)^{-3} && \text{apply } \left(\frac{a}{b}\right)^{n}=\frac{a^n}{b^n}\\
&= \frac{(-3)^2}{\left(2a^{-6}b^{7}\right)^{2}}\left(a^3b^{-5}\right)^{-3} && \text{apply } (ab)^n=a^nb^n \text{ and } a^{-n}=\frac{1}{a^n}\\
&= \frac{9}{2^2\left(a^{-6}\right)^2\left(b^7\right)^2}\cdot\frac{1}{\left(a^3b^{-5}\right)^3} && \text{apply } (a^n)^m=a^{nm} \text{ and } (ab)^n=a^nb^n\\
&= \frac{9}{4a^{-12}b^{14}}\cdot\frac{1}{\left(a^3\right)^3\left(b^{-5}\right)^3} && \text{apply } a^{-n}=\frac{1}{a^n} \text{ and } (ab)^n=a^nb^n\\
&= \frac{9a^{12}}{4b^{14}}\cdot\frac{1}{a^{3\cdot 3}b^{(-5)3}}\\
&= \frac{9a^{12}}{4b^{14}}\cdot\frac{1}{a^{9}b^{-15}} && \text{apply } a^{-n}=\frac{1}{a^n}\\
&= \frac{9a^{12}}{4b^{14}}\cdot\frac{b^{15}}{a^{9}} = \frac{9a^{12}b^{15}}{4b^{14}a^9} && \text{apply } \frac{a^n}{a^m}=a^{n-m}\\
&= \frac{9a^{12-9}b^{15-14}}{4} = \frac{9a^3b^1}{4} = \frac{9}{4}a^3b
\end{aligned}$$

27. $\left(-2a^{-3}\right)\left(-2a^{-2}b\right)^{-4}$

Solution:

$$\begin{aligned}
E &= \left(-2a^{-3}\right)\left(-2a^{-2}b\right)^{-4} && \text{apply } a^{-n}=\frac{1}{a^n}\\
&= \left(-2\cdot\frac{1}{a^3}\right)\frac{1}{\left(-2a^{-2}b\right)^4} && \text{apply } (ab)^n=a^nb^n\\
&= \left(\frac{-2}{1}\cdot\frac{1}{a^3}\right)\frac{1}{(-2)^4\left(a^{-2}\right)^4b^4} && \text{apply } (a^n)^m=a^{nm}\\
&= \frac{-2}{a^3}\cdot\frac{1}{16a^{-8}b^4} && \text{apply } a^{-n}=\frac{1}{a^n}\\
&= \frac{-2}{a^3}\cdot\frac{a^8}{16b^4}\\
&= \frac{-2a^8}{a^3\cdot 16b^4} = \frac{-2a^8}{16a^3b^4} && \text{apply } \frac{a^n}{a^m}=a^{n-m}\\
&= \frac{-1\cdot\cancel{2}a^{8-3}}{8\cdot\cancel{2}b^4} = \frac{-a^5}{8b^4}
\end{aligned}$$

28. $\dfrac{\left(-3p^3q^5\right)^2}{\left(2q^0p^3\right)^{-1}}$

Solution:

$$\begin{aligned}
E &= \frac{\left(-3p^3q^5\right)^2}{\left(2q^0p^3\right)^{-1}} && \text{apply } q^0 = 1 \text{ and } \frac{1}{a^{-n}} = a^n \\
&= \left(-3p^3q^5\right)^2\left(2\cdot 1p^3\right)^1 \\
&= \left(-3p^3q^5\right)^2 \cdot 2p^3 && \text{apply } (ab)^n = a^nb^n \\
&= (-3)^2\left(p^3\right)^2\left(q^5\right)^2 \cdot 2p^3 && \text{apply } (a^n)^m = a^{nm} \\
&= 9p^{3\cdot 2}q^{5\cdot 2}\cdot 2p^3 \\
&= 18p^6q^{10}p^3 && \text{apply } a^n\cdot a^m = a^{n+m} \\
&= 18p^{6+3}q^{10} = 18p^9q^{10}
\end{aligned}$$

29. $\left(\dfrac{2a^{-2}b^3}{-2^2\left(a^{-1}b\right)^{-3}}\right)^{-2}$

Solution:

$$\begin{aligned}
E &= \left(\frac{2a^{-2}b^3}{-2^2\left(a^{-1}b\right)^{-3}}\right)^{-2} && \text{apply } \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \\
&= \left(\frac{-2^2\left(a^{-1}b\right)^{-3}}{2a^{-2}b^3}\right)^2 && \text{apply } (ab)^n = a^nb^n \\
&= \left(\frac{-4\left(a^{-1}\right)^{-3}b^{-3}}{2a^{-2}b^3}\right)^2 && \text{apply } (a^n)^m = a^{nm} \\
&= \left(\frac{-4a^{-1(-3)}b^{-3}}{2a^{-2}b^3}\right)^2 \\
&= \left(\frac{-2a^3b^{-3}}{a^{-2}b^3}\right)^2 && \text{apply } \frac{a^n}{a^m} = a^{n-m} \\
&= \left(-2a^{3-(-2)}b^{-3-3}\right)^2 \\
&= \left(-2a^{3+2}b^{-3-3}\right)^2 \\
&= \left(-2a^5b^{-6}\right)^2 && \text{apply } (ab)^n = a^nb^n \\
&= (-2)^2\left(a^5\right)^2\left(b^{-6}\right)^2 && \text{apply } (a^n)^m = a^{nm} \\
&= 4a^{5\cdot 2}b^{-6\cdot 2} = 4a^{10}b^{-12} && \text{apply } a^{-n} = \frac{1}{a^n} \\
&= 4a^{10}\cdot\frac{1}{b^{12}} \\
&= \frac{4a^{10}}{1}\cdot\frac{1}{b^{12}} = \frac{4a^{10}}{b^{12}}
\end{aligned}$$

30. $\left(-\dfrac{x^3y^0x^{-5}}{y^{-3}}\right)^{-2}$

Solution:

$$\begin{aligned}
E &= \left(-\frac{x^3y^0x^{-5}}{y^{-3}}\right)^{-2} && y^0=1 \text{ and } a^n\cdot a^m=a^{n+m}\\
&= \left(-\frac{x^{3+(-5)}}{y^{-3}}\right)^{-2} = \left(\frac{-1x^{-2}}{y^{-3}}\right)^{-2} && \text{apply } \left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}\\
&= \frac{\left(-1x^{-2}\right)^{-2}}{\left(y^{-3}\right)^{-2}} && \text{apply } (ab)^n=a^nb^n\\
&= \frac{(-1)^{-2}\left(x^{-2}\right)^{-2}}{\left(y^{-3}\right)^{-2}} && \text{apply } (a^n)^m=a^{nm} \text{ and } a^{-n}=\frac{1}{a^n}\\
&= \frac{x^{-2(-2)}}{(-1)^2\,y^{-3(-2)}} = \frac{x^4}{1y^6} = \frac{x^4}{y^6}
\end{aligned}$$

31. $\left(-\dfrac{x^3y^7x^{-5}}{y^{-3}}\right)^{0}$

Solution: Any non-zero quantity raised to the power zero is $1$. So the answer is $1$.

32. $\dfrac{x^{-1}+y^{-1}}{x^{-2}-y^{-2}}$

Solution: This problem is very different because there are addition and subtraction involved. Because of that, we can not simply move the expressions with negative exponents. Instead, this will be a problem involving complex fractions.

$$\begin{aligned}
E &= \frac{x^{-1}+y^{-1}}{x^{-2}-y^{-2}}\\
&= \frac{\dfrac{1}{x^1}+\dfrac{1}{y^1}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}}\\
&= \frac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}} && \text{bring fractions to the common denominator}\\
&= \frac{\dfrac{1\cdot y}{x\cdot y}+\dfrac{1\cdot x}{y\cdot x}}{\dfrac{1\cdot y^2}{x^2\cdot y^2}-\dfrac{1\cdot x^2}{y^2\cdot x^2}} = \frac{\dfrac{y}{xy}+\dfrac{x}{xy}}{\dfrac{y^2}{x^2y^2}-\dfrac{x^2}{x^2y^2}}\\
&= \frac{\dfrac{y+x}{xy}}{\dfrac{y^2-x^2}{x^2y^2}} && \text{to divide is to multiply by the reciprocal}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{y+x}{xy}\cdot\frac{x^2y^2}{y^2-x^2} && \text{cancel out } xy \\
&= \frac{y+x}{1}\cdot\frac{xy}{y^2-x^2} \\
&= \frac{xy\,(x+y)}{y^2-x^2} && \text{factor } y^2-x^2 \text{ via the difference of} \\
&= \frac{xy\,(x+y)}{(y-x)\,(y+x)} && \text{squares theorem, cancel out } x+y \\
&= \frac{xy}{y-x}
\end{aligned}
$$

33. $\dfrac{\left(-2a^{-2}\right)^{-2}b^3a^0\left(-aba^{-2}b^{-2}\right)^{-3}}{2a^2\left(-2a^{-2}b\right)^{-2}ab^0}$

Solution:

$$
\begin{aligned}
E &= \frac{\left(-2a^{-2}\right)^{-2}b^3a^0\left(-aba^{-2}b^{-2}\right)^{-3}}{2a^2\left(-2a^{-2}b\right)^{-2}ab^0} && a^0=b^0=1 \text{ and } x^nx^m=x^{n+m} \\
&= \frac{\left(-2a^{-2}\right)^{-2}b^3\left(-a^{1+(-2)}b^{1+(-2)}\right)^{-3}}{2a^{2+1}\left(-2a^{-2}b\right)^{-2}} \\
&= \frac{\left(-2a^{-2}\right)^{-2}b^3\left(-1a^{-1}b^{-1}\right)^{-3}}{2a^3\left(-2a^{-2}b\right)^{-2}} && \text{apply } (xy)^n=x^ny^n \\
&= \frac{(-2)^{-2}\left(a^{-2}\right)^{-2}b^3(-1)^{-3}\left(a^{-1}\right)^{-3}\left(b^{-1}\right)^{-3}}{2a^3(-2)^{-2}\left(a^{-2}\right)^{-2}b^{-2}} && \text{apply } (x^n)^m=x^{nm} \\
&= \frac{(-2)^{-2}a^{-2(-2)}b^3(-1)^{-3}a^{-1(-3)}b^{-1(-3)}}{2a^3(-2)^{-2}a^{-2(-2)}b^{-2}} \\
&= \frac{(-2)^{-2}a^4b^3(-1)^{-3}a^3b^3}{2a^3(-2)^{-2}a^4b^{-2}} && \text{cancel out } a^4 \text{ and } a^3 \text{ and } (-2)^{-2} \\
&= \frac{b^3(-1)^{-3}b^3}{2b^{-2}} && \text{apply } x^nx^m=x^{n+m} \\
&= \frac{(-1)^{-3}b^{3+3}}{2b^{-2}} \\
&= \frac{(-1)^{-3}b^6}{2b^{-2}} && \text{apply } x^{-n}=\frac{1}{x^n} \\
&= \frac{b^6b^2}{(-1)^3\,2} && \text{apply } x^nx^m=x^{n+m} \\
&= \frac{b^{6+2}}{-1\cdot 2}=\frac{b^8}{-2}=-\frac{b^8}{2}
\end{aligned}
$$

34. $\left(\dfrac{-a^2\left(b^{-1}a\right)^{-5}}{b^7\left(-ab^2\right)^{-3}}\right)^{-2}$

Solution:

$$\begin{aligned}
E &= \left(\frac{-a^2\left(b^{-1}a\right)^{-5}}{b^7\left(-ab^2\right)^{-3}}\right)^{-2} && \text{apply } (xy)^n = x^n y^n \\
&= \left(\frac{-a^2\left(b^{-1}\right)^{-5}a^{-5}}{b^7\left(-1\right)^{-3}a^{-3}\left(b^2\right)^{-3}}\right)^{-2} && \text{apply } (x^n)^m = x^{nm} \\
&= \left(\frac{-a^2b^{-1(-5)}a^{-5}}{b^7\left(-1\right)^{-3}a^{-3}b^{2(-3)}}\right)^{-2} = \left(\frac{-a^2b^5a^{-5}}{b^7\left(-1\right)^{-3}a^{-3}b^{-6}}\right)^{-2} && \text{apply } x^n x^m = x^{n+m} \\
&= \left(\frac{-a^{2+(-5)}b^5}{\left(-1\right)^{-3}b^{7+(-6)}a^{-3}}\right)^{-2} = \left(\frac{-1\cdot a^{-3}b^5}{\left(-1\right)^{-3}b^1a^{-3}}\right)^{-2} && \text{cancel out } a^{-3} \\
&= \left(\frac{-1b^5}{\left(-1\right)^{-3}b^1}\right)^{-2} && \text{apply } a^{-n} = \frac{1}{a^n} \\
&= \left(\frac{-1\left(-1\right)^3b^5}{b^1}\right)^{-2} = \left(\frac{-1\left(-1\right)b^5}{b^1}\right)^{-2} = \left(\frac{1b^5}{b^1}\right)^{-2} && \text{apply } \frac{x^n}{x^m} = x^{n-m} \\
&= \left(b^{5-1}\right)^{-2} = \left(b^4\right)^{-2} && \text{apply } (x^n)^m = x^{nm} \\
&= b^{4(-2)} = b^{-8} = \frac{1}{b^8}
\end{aligned}$$

35. $\dfrac{\left(x^{-2}\right)^{-2}y^3x^0\left(-2yx^0y^{-2}x^{-2}\right)^0}{yx^5\left(y^{-2}x\right)^{-3}\left(2x^{-1}yx^3\right)^{-1}}$

Solution:

$$\begin{aligned}
E &= \frac{\left(x^{-2}\right)^{-2}y^3x^0\left(-2yx^0y^{-2}x^{-2}\right)^0}{yx^5\left(y^{-2}x\right)^{-3}\left(2x^{-1}yx^3\right)^{-1}} && \text{apply } a^0 = 1 \text{ and } a^n a^m = a^{n+m} \\
&= \frac{\left(x^{-2}\right)^{-2}y^3}{yx^5\left(y^{-2}x\right)^{-3}\left(2x^{-1+3}y\right)^{-1}} && \text{apply } (ab)^n = a^n b^n \\
&= \frac{\left(x^{-2}\right)^{-2}y^3}{yx^5\left(y^{-2}\right)^{-3}x^{-3}\left(2x^2y\right)^{-1}} && \text{apply } (ab)^n = a^n b^n \text{ and } a^n a^m = a^{n+m} \\
&= \frac{\left(x^{-2}\right)^{-2}y^3}{yx^{5+(-3)}\left(y^{-2}\right)^{-3}\left(2\right)^{-1}\left(x^2\right)^{-1}y^{-1}} && \text{apply } (a^n)^m = a^{nm} \\
&= \frac{x^{-2(-2)}y^3}{yx^2y^{-2(-3)}2^{-1}x^{2(-1)}y^{-1}} = \frac{x^4y^3}{2^{-1}yx^2y^6x^{-2}y^{-1}} && \text{apply } a^n a^m = a^{n+m} \\
&= \frac{x^4y^3}{2^{-1}y^{1+6+(-1)}x^{2+(-2)}} = \frac{x^4y^3}{2^{-1}y^6x^0} && x^0 = 1 \text{ and } \frac{a^n}{a^m} = a^{n-m} \\
&= \frac{x^4y^{3-6}}{2^{-1}} = \frac{x^4y^{-3}}{2^{-1}} && \text{apply } a^{-n} = \frac{1}{a^n} \\
&= \frac{2^1x^4}{y^3} = \frac{2x^4}{y^3}
\end{aligned}$$

