Definition: To **factor** something means to re-write it as a product.

Factoring will be a very important step in solving many types of problems. Most importantly, factoring is key in solving equations of degree 2 (also called *quadratic*), degree 3 (also called *cubic*), degree 4, and so on. This is because of the zero product rule. Let us recall this rule first.

Theorem: Suppose that we multiply some numbers and the result is zero.
Then:
1.) One of the factors must be zero, and
2.) the values of all other factors are irrelevant.

This property is only true for zero. Suppose that the product of two numbers is 100. The value of the two factors depend on each other. Let's say we start with $1 \cdot 100$. If we increase the first factor, the second factor must decrease, as in $2 \cdot 50$ or $5 \cdot 20$. It is a balancing act. Only zero has the very special property that allows us to focus on only one factor while ignoring all other factors.

For example, the zero product rule can be used to solve the equation $(x+3)(x-1)=0$. If two factors multiply to zero, one of the factors must be zero. So, there are only two possibilities: either $x+3=0$ (and we don't need to worry about the second factor), or $x-1=0$ (and we don't need to worry about the value of the first factor.) The zero product rule allowed us to trade in one quadratic (of degree 2) equation for two linear equations: $x+3=0$ and $x-1=0$. We solve these equations and obtain -3 and 1 as solution.

Equations with degree 2, 3, 4, 5, and beyond can be solved by the zero product rule. So, if an equation is of a degree higher than 1, we will reduce one side to zero, factor the other side and apply the zero product rule. For this reason, factoring algebraic expressions is a very important task.

There are many factoring techniques, and we will learn many of them. Different techniques work on different expressions. The process of factoring starts with inspecting the expression to decide which techiques would work. There is one exception to this: in all cases, our first step must be **factoring out the greatest common factor**. We will see later examples in which the additional techniques can not even be applied unless we factor out the greatest common factor or GCF first.

Recall the distributive law:

Axiom (The Distributive Law): For all real numbers a, b, and c,

$$a(b+c)=ab+ac$$

Consider the expression $2(5x-9)$. We can apply the distributive law to expand this expression:

$$2(5x-9)=10x-18$$

Factoring out the greatest common factor is the reversal of this process.

Example 1. Factor out the greatest common factor in $12x - 18$.

Solution: The first step is to identify the greatest common factor or GCF. Both $12x$ and -18 are divisible by 6.

We write $6\,(\qquad)$ and the rest is a few division problems.

We ask: 6 times what will give us $12x$? The answer is $2x$ because $6 \cdot 2x = 12x$. Similarly, 6 times what will give us -18? The answer is -3. We can now write:

$$12x - 18 = \boxed{6\,(2x-3)}$$

After we wrote down what we think the answer is, we need to ask two questions. Does the multiplication backward work? Did we get all common divisors out? We distribute 6 in $6\,(2x-3)$ and see that we get the correct product. If we inspect $2x-3$, we see that the two terms do not share any divisors, and so we did factor out the greatest common factor.

Example 2. Factor out the greatest common factor in $10a^3b^2 - 5ab + 30ab^3$.

Solution: We first identify the greatest common factor between the three terms in $10a^3b^2 - 5ab + 30ab^3$. The numbers multiplying the variables, also called coefficients are $10, -5$, and 30. Their greatest common factor is 5. Then we look for $a-$powers. The first term is divisible by a^3, the second term by a, and the third term by a. The greatest common factor between them is a. Similarly, the greatest common factor of b^2, b, and b^3 is b. Therefore, the greatest common factor is $5ab$. So we write $5ab\,(\qquad)$ and the rest is three division problems.

$$10a^3b^2 - 5ab + 30ab^3 = 5ab\,(\qquad)$$

We will need to write three terms into the parentheses. In case of all factoring, we usually ask: does the multiplication backward work? $5ab$ must be multplied by what, so that the product is $10a^3b^2$. The answer is $2a^2b$. So now we have:

$$10a^3b^2 - 5ab + 30ab^3 = 5ab\left(2a^2b \qquad\right)$$

Once we wrote down the first term, we can check whether the multiplication backwards work. For the second term, $-5ab$, nearly everything was factored out. If this happens, we are left with 1. In this case, we are left with -1.

$$10a^3b^2 - 5ab + 30ab^3 = 5ab\left(2a^2b - 1 \qquad\right)$$

For the third term, we ask: $5ab$ times what is $30ab^3$? The answer is $6b^2$, and so we have

$$10a^3b^2 - 5ab + 30ab^3 = \boxed{5ab\left(2a^2b - 1 + 6b^2\right)}$$

We ask the two questions. *Does the multiplication backward work?* and *Did we get all the common factors out?* Applying the distributive law, we see that the multiplication backward does work. Inspecting the three terms inside the parentheses, we see that they do not share any divisors. This is especially easy, given that the second term is -1. Thus our solution is correct.

Sometimes we will need to factor out -1 from an expression. This step is usually needed when the coefficient of the highest degree term is -1.

Example 3. Factor out -1 from $8x^5 - x^6 + 3x - 2$.

Solution: It is always a good idea to rearrange the terms by degree. Then we write $-1\,(\qquad)$. Inside the parentheses, we write the opposite of our expression, i.e. change all signs.

$$8x^5 - x^6 + 3x - 2 = -x^6 + 8x^5 + 3x - 2 = \boxed{-1\left(x^6 - 8x^5 - 3x + 2\right)}$$

We often omit the 1 and write only $-\left(x^6 - 8x^5 - 3x + 2\right)$.

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Sometimes the greatest common factor is more complicated.

Example 4. Factor out the GCF from $12a^3(a-2) - 6a^2(a-2) + 24(a-2)$.

Solution: In this case, $a-2$ is part of the GCF. We factor it out:

$$12a^3(a-2) - 6a^2(a-2) + 24(a-2) = (a-2)\left(12a^3 - 6a^2 + 24\right)$$

If we look at the expression in the second pair of parentheses, we see that there is a common factor of 6. Thus the final answer is

$$(a-2)\,6\left(2a^3 - a^2 + 4\right) = \boxed{6(a-2)\left(2a^3 - a^2 + 4\right)}$$

Factoring out the GCF must always be the first step in factoring. In case of the next example, this is all we need.

Example 5. Solve the equation $x^2 = 6x$

Solution: We realize that this is a quadratic equation. Therefore, we need to reduce one side to zero, factor, and apply the zero product rule. The number multiplying the variables in the highest degree term is called **the leading coefficient**. When reducing one side to zero, we should try to avoid creating negative leading coefficients. In this case, we should subtract $6x$ from both sides.

$$\begin{aligned} x^2 &= 6x && \text{subtract } 6x \\ x^2 - 6x &= 0 && \text{factor out the GCF} \\ x(x-6) &= 0 && \end{aligned}$$

We apply the zero product rule to the two factors:

$$x = 0 \quad \text{or} \quad x - 6 = 0$$
$$x = 6$$

Therefore, there are two solutions, $\boxed{0 \text{ and } 6}$. We check: if $x = 0$, then both sides are zero. If $x = 6$, then both sides are 36. Thus our solution is correct.

Example 6. Find all numbers with the following property. The number raised to the third power is five times the number we get if we double the number and then square the result.

Solution: We label this number by x. Then the number raised to the third power is x^3. If we double the number, we get $2x$. We write the equation comparing the square of $2x$ and x^3.

$$\begin{aligned} 5\left((2x)^2\right) &= x^3 && \\ 5\left(4x^2\right) &= x^3 && \\ 20x^2 &= x^3 && \text{subtract } 20x^2 \\ 0 &= x^3 - 20x^2 && \text{factor out the GCF} \\ 0 &= x^2(x-20) && \text{apply the zero product rule} \end{aligned}$$

$$x = 0 \;\text{ or }\; x = 20$$

So there are two such numbers: $\boxed{0 \text{ and } 20}$. We check: 0 clearly works. If the number is 20, it raised to the third power is $20^3 = 8000$. If we double 20, we get 40. The square of 40 is $40^2 = 1600$, and indeed 8000 is five times 1600, thus our solution is correct.

Sample Problems

1. Factor out the greatest common factor in each of the given expressions.

   a) $3x - 12$
   b) $16a^2b + 20a^3b - 12a^2b^2$
   c) $3a^2 - 12$
   d) $3a^3 - 12a^2$
   e) $20x + 5x^3$
   f) $3x(x-2) + 8x^3(x-2) - 11(x-2)$

2. Factor out -1 from $-5x^3 + 2x^2 - x - 8$.

3. Solve each of the following equations. Make sure to check your solution.

   a) $(x-2)(x+3)(2x+1) = 0$ b) $m(m+7) = 0$ c) $x^2 = 9x$ d) $8x^3 = 50x^2$

4. Find all numbers that satisfy the following condition: if we square the number, we get back the same number.

Practice Problems

1. Factor out the greatest common factor from each of the following.

   a) $10a^2b^2 - 15ab^3 + 25a^2b^3c$
   b) $6x^3 - 3x^2 - 15x^4$
   c) $a^2 - a^3 + a^4$
   d) $6a^2b + 12a^3b - 30a^3b^2$
   e) $x^5 - 2x^4 + 4x^3$
   f) $3xy(a-3) + 8t(a-3) - 200x^5(a-3)$

2. Factor out -1 from each of the following.

   a) $x^3 - x^5 + 2$ b) $-x^2 + 3x - 1$ c) $-x^2 + 3x - 5$

3. Solve each of the following equations. Make sure to check your solutions.

   a) $(w+5)(w-1) = 0$
   b) $x(x-2)(x+3) = 0$
   c) $2(x-2)(x+3) = 0$
   d) $x^2 = 4x$
   e) $x^2 + 6x = 0$
   f) $3x^3 = 75x^2$

4. A number has the following property: if we square it, we obtain the opposite of the number. Find all such numbers.

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Answers

Sample Problems

1. a) $3(x-4)$ b) $4a^2b(5a-3b+4)$ c) $3(a^2-4)$ d) $3a^2(a-4)$ e) $5x(x^2+4)$ f) $(x-2)(8x^3+3x-11)$

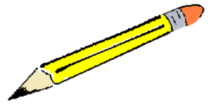
2. $-(5x^3-2x^2+x+8)$ 3. a) $2, -3$, and $-\frac{1}{2}$ b) 0 and -7 c) 0 and 9 d) 0 and $\frac{25}{4}$ 4. $0, 1$

Practice Problems

1. a) $5ab^2(2a-3b+5abc)$ b) $3x^2(2x-5x^2-1)$ c) $a^2(a^2-a+1)$ d) $6a^2b(2a-5ab+1)$
   e) $x^3(x^2-2x+4)$ f) $(a-3)(3xy+8t-200x^5)$

2. a) $-(-x^3+x^5-2)$ b) $-(x^2-3x+1)$ c) $-(x^2-3x+5)$

3. a) $-5, 1$ b) $0, 2, -3$ c) $2, -3$ d) $0, 4$ e) $0, -6$ f) $0, 25$ 4. $0, -1$

Sample Problems  Solutions

1. Completely factor each of the following.

a) $3x - 12$
Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 3.

$$3x - 12 = \boxed{3(x-4)}$$

What is in the parentheses, $x - 4$ can not be further factored. We can easily check our work by multiplication.

b) $16a^2b + 20a^3b - 12a^2b^2$
Solution: First we identify the GCF (greatest common factor). In this case, the GCF is $4a^2b$. So we have

$$4a^2b\,(\qquad\qquad)$$

and need to figure out what to write into the parenthese so that the multiplication backwards works. What do we have to multiply $4a^2b$ by so the result is $16a^2b$? The answer is $\dfrac{16a^2b}{4a^2b} = 4$, so we have so far

$$4a^2b\,(4\qquad\qquad)$$

Next, what do we have to multiply $4a^2b$ by so the result is $20a^3b$? The answer is $\dfrac{20a^3b}{4a^2b} = 5a$ and so now we have

$$4a^2b\,(4+5a\qquad)$$

Next, what do we have to multiply $4a^2b$ by so the result is $-12a^2b^2$? The answer is $\dfrac{-12a^2b^2}{4a^2b} = -3b$ and so now we have

$$\boxed{4a^2b\,(4+5a-3b)}$$

We check via multiplication backwards:

$$\begin{aligned} 4a^2b\,(4+5a-3b) &= 4a^2b\,(4) + 4a^2b\,(5a) + 4a^2b\,(-3b) \\ &= 16a^2b + 20a^3b - 12a^2b^2 \end{aligned}$$

and so our solution is correct.

c) $3a^2 - 12$
Solution: The greatest common factor (or GCF) is 3.

$$3a^2 - 12 = 3\left(a^2-4\right)$$

We check our work by multiplication:

$$3\left(a^2-4\right) = 3a^2 - 12$$

and so our answer, $\boxed{3\left(a^2-4\right)}$ is correct.

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d) $3a^3 - 12a^2$
Solution: The greatest common factor (or GCF) is $3a^2$.

$$3a^3 - 12a^2 = 3a^2 \left(\qquad \right)$$

what do we have to multiply $3a^2$ by so the result is $3a^3$? The answer is $\dfrac{3a^3}{3a^2} = a$ and so now we have

$$3a^3 - 12a^2 = 3a^2 \left(a \qquad \right)$$

Next, what do we have to multiply $3a^2$ by so the result is $-12a^2$? The answer is $\dfrac{-12a^2}{3a^2} = -4$ and so now we have

$$3a^3 - 12a^2 = 3a^2 \left(a - 4 \right)$$

We check our work by multiplication:

$$3a^2 \left(a - 4 \right) = 3a^2 \cdot a - 3a^2 \cdot 4 = 3a^3 - 12a^2$$

and so our answer, $\boxed{3a^2 \left(a - 4 \right)}$ is correct.

e) $20x + 5x^3$
Solution: We rearrange the terms by degree first and then factor out the GCF.

$$20x + 5x^3 = 5x^3 + 20x = 5x \left(x^2 + 4 \right)$$

The final answer is $\boxed{5x \left(x^2 + 4 \right)}$. We can easily check the result by mulitplication.

f) $3x \left(x - 2 \right) + 8x^3 \left(x - 2 \right) - 11 \left(x - 2 \right)$
Solution: Now the GCF is the linear expression $x - 2$. So we factor ot out:

$$\left(x - 2 \right) \left(\qquad \right)$$

What do we have to multiply $x - 2$ by so the result is $3x \left(x - 2 \right)$? The answer is $3x$ and so now we have

$$\left(x - 2 \right) \left(3x \qquad \right)$$

Next, what do we have to multiply $x - 2$ by so the result is $8x^3 \left(x - 2 \right)$? The answer is $8x^3$ and so now we have

$$\left(x - 2 \right) \left(3x + 8x^3 \qquad \right)$$

Next, what do we have to multiply $x - 2$ by so the result is $-11 \left(x - 2 \right)$? The answer is -11 and so now we have

$$\left(x - 2 \right) \left(3x + 8x^3 - 11 \right)$$

It is good practice to rearrange polynomials by degree and so our answer is $\boxed{\left(x - 2 \right) \left(8x^3 + 3x - 11 \right)}$.

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2. Factor out -1 from the given expression.

$$-5x^3+2x^2-x-8$$

Solution: We start with a minus sign (short for -1) and a parentheses.

$$-5x^3+2x^2-x-8=-(\qquad\qquad)$$

Inside the parenthese, we write the original expression, but change all signs. Again, when we multiply back to check and apply the distributive law, we should get back the original expression.

$$-5x^3+2x^2-x-8=-\left(5x^3-2x^2+x+8\right)$$

So our answer is $\boxed{-\left(5x^3-2x^2+x+8\right)}$.

3. Solve each of the following equations. Make sure to check your solution.

a) $(x-2)(x+3)(2x+1)=0$
Solution: Since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule. Most of these were already done for us as the right-hand side is zero and the left-hand side is completely factored. All we need to do is apply the zero product rule. **A product can only be zero if one of its factors is zero.** $(x-2)(x+3)(2x+1)=0$ means that either $x-2=0$ or $x+3=0$ or $2x+1=0$. We solve these linear equations separately:

$$\begin{array}{ccccc} x-2=0 & \text{or} & x+3=0 & \text{or} & 2x+1=0 \\ x=2 & & x=-3 & & 2x=-1 \\ & & & & x=-\dfrac{1}{2} \end{array}$$

We check all three solutions. If $x=2$, then

$$(2-2)(2+3)(2(2)+1)=0\cdot5\cdot5=0$$

If $x=-3$, then

$$(-3-2)(-3+3)(2(-3)+1)=-5\cdot0\cdot(-5)=0$$

and if $x=-\frac{1}{2}$, then

$$\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}+3\right)\left(2\left(-\frac{1}{2}\right)+1\right)=-\frac{3}{2}\cdot\frac{5}{2}\cdot0=0$$

and so all three numbers, $\boxed{2,\ -3,\ \text{and}\ -\frac{1}{2}}$ are correct.

b) $m(m+7)=0$

Solution: We will apply the zero product rule. **A product can only be zero if one of its factors is zero.** $m(m+7)=0$ means that either $m=0$. We solve these linear equations separately and obtain $m=0$ and $m=-7$. We check: If $m=0$, then

$$0(0+7)=0\cdot7=0$$

and if $m=-7$, then

$$-7(-7+7)=-7\cdot0$$

and so both numbers, $\boxed{0\text{ and }-7}$ are correct.

c) $x^2 = 9x$

Solution: Since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} x^2 &= 9x && \text{subtract } 9x \\ x^2 - 9x &= 0 && \text{factor out the GCF} \\ x(x-9) &= 0 \end{aligned}$$

A product can only be zero if one of its factors is zero. $x(x-9) = 0$ means that either $x = 0$ or $x - 9 = 0$. We solve these linear equations separately and obtain $\boxed{0 \text{ and } 9}$. We check: $0^2 = 9 \cdot 0$ and $9^2 = 9 \cdot 9$ and so our solution is correct.

d) $8x^3 = 50x^2$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} 8x^3 &= 50x^2 && \text{subtract } 50x^2 \\ 8x^3 - 50x^2 &= 0 && \text{the GCF is } 2x^2 \\ 2x^2(4x-25) &= 0 \end{aligned}$$

We now apply the zero product rule. If this product is zero, then either $2x^2 = 0$ or $4x - 25 = 0$. We solve these equations for x.

$$\begin{aligned} 2x^2 &= 0 && \text{or} && 4x - 25 = 0 \\ 2 \cdot x \cdot x &= 0 && \text{or} && 4x = 25 \\ x &= 0 && \text{or} && x = \frac{25}{4} \end{aligned}$$

We check both solutions. If $x = 0$, then LHS$= 8 \cdot 0^3 = 8 \cdot 0 = 0$ and RHS$= 50 \cdot 0^2 = 50 \cdot 0 = 0$

If $x = \frac{25}{4}$, then

$$\text{LHS} = 8\left(\frac{25}{4}\right)^3 = \frac{8}{1} \cdot \frac{15\,625}{64} = \frac{15\,625}{8} \quad \text{and} \quad \text{RHS} = 50\left(\frac{25}{4}\right)^2 = \frac{50}{1} \cdot \frac{625}{16} = \frac{15\,625}{8}$$

Thus both solutions, $\boxed{0 \text{ and } \frac{25}{4}}$ are correct.

4. Find all numbers that satisfy the following condition: if we square the number, we get back the same number.

Solution: Let us denote the number by x. The equation is

$$\begin{aligned} x^2 &= x && \text{reduce one side to zero} \\ x^2 - x &= 0 && \text{factor} \\ x(x-1) &= 0 && \text{apply the zero property} \\ x = 0 \quad &\text{or} \quad x - 1 = 0 \\ x = 0 \quad &\text{or} \quad x = 1 \end{aligned}$$

Thus there are two numbers, 0 and 1, satisfying the property. We check: $0^2 = 0$ and $1^2 = 1$. Thus our answer is: $\boxed{0 \text{ and } 1}$.

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