Part 1 – Factoring out the GCF

Definition: To **factor** something means to re-write it as a product.

Factoring will be a very important step in solving many types of problems. Most importantly, factoring is key in solving equations of degree 2 (also called *quadratic*), degree 3 (also called *cubic*), degree 4, and so on. This is because of the zero product rule. Let us recall this rule first.

Theorem: Suppose that we multiply some numbers and the result is zero.
Then:
1.) One of the factors must be zero, and
2.) the values of all other factors are irrelevant.

This property is only true for zero. Suppose that the product of two numbers is 100. The value of the two factors depend on each other. Let's say we start with $1 \cdot 100$. If we increase the first factor, the second factor must decrease, as in $2 \cdot 50$ or $5 \cdot 20$. It is a balancing act. Only zero has the very special property that allows us to focus on only one factor while ignoring all other factors.

For example, the zero product rule can be used to solve the equation $(x+3)(x-1) = 0$. If two factors multiply to zero, one of the factors must be zero. So, there are only two possibilities: either $x + 3 = 0$ (and we don't need to worry about the second factor), or $x - 1 = 0$ (and we don't need to worry about the value of the first factor.) The zero product rule allowed us to trade in one quadratic (of degree 2) equation for two linear equations: $x + 3 = 0$ and $x - 1 = 0$. We solve these equations and obtain -3 and 1 as solution.

Equations with degree 2, 3, 4, 5, and beyond can be solved by the zero product rule. So, if an equation is of a degree higher than 1, we will reduce one side to zero, factor the other side and apply the zero product rule. For this reason, factoring algebraic expressions is a very important task.

There are many factoring techniques, and we will learn many of them. Different techniques work on different expressions. The process of factoring starts with inspecting the expression to decide which techiques would work. There is one exception to this: in all cases, our first step must be **factoring out the greatest common factor**. We will see later examples in which the additional techniques can not even be applied unless we factor out the greatest common factor or GCF first.

Recall the distributive law:

Axiom (The Distributive Law): For all real numbers a, b, and c,

$$a(b+c) = ab + ac$$

Consider the expression $2(5x-9)$. We can apply the distributive law to expand this expression:

$$2(5x-9) = 10x - 18$$

Factoring out the greatest common factor is the reversal of this process.

Example 1. Factor out the greatest common factor in $12x - 18$.

Solution: The first step is to identify the greatest common factor or GCF. Both $12x$ and -18 are divisible by 6.

We write $6\left(\qquad \right)$ and the rest is a few division problems.

We ask: 6 times what will give us $12x$? The answer is $2x$ because $6 \cdot 2x = 12x$. Similarly, 6 times what will give us -18? The answer is -3. We can now write:

$$12x - 18 = \boxed{6\left(2x - 3\right)}$$

After we wrote down what we think the answer is, we need to ask two questions. Does the multiplication backward work? Did we get all common divisors out? We distribute 6 in $6\left(2x - 3\right)$ and see that we get the correct product. If we inspect $2x - 3$, we see that the two terms do not share any divisors, and so we did factor out the greatest common factor.

Example 2. Factor out the greatest common factor in $10a^3b^2 - 5ab + 30ab^3$.

Solution: We first identify the greatest common factor between the three terms in $10a^3b^2 - 5ab + 30ab^3$. The numbers multiplying the variables, also called coefficients are $10, -5$, and 30. Their greatest common factor is 5. Then we look for $a-$powers. The first term is divisible by a^3, the second term by a, and the third term by a. The greatest common factor between them is a. Similarly, the greatest common factor of b^2, b, and b^3 is b. Therefore, the greatest common factor is $5ab$. So we write $5ab\left(\qquad \right)$ and the rest is three division problems.

$$10a^3b^2 - 5ab + 30ab^3 = 5ab\left(\qquad \right)$$

We will need to write three terms into the parentheses. In case of all factoring, we usually ask: does the multiplication backward work? $5ab$ must be multplied by what, so that the product is $10a^3b^2$. The answer is $2a^2b$. So now we have:

$$10a^3b^2 - 5ab + 30ab^3 = 5ab\left(2a^2b \qquad \right)$$

Once we wrote down the first term, we can check whether the multiplication backwards work. For the second term, $-5ab$, nearly everything was factored out. If this happens, we are left with 1. In this case, we are left with -1.

$$10a^3b^2 - 5ab + 30ab^3 = 5ab\left(2a^2b - 1 \qquad \right)$$

For the third term, we ask: $5ab$ times what is $30ab^3$? The answer is $6b^2$, and so we have

$$10a^3b^2 - 5ab + 30ab^3 = \boxed{5ab\left(2a^2b - 1 + 6b^2\right)}$$

We ask the two questions. *Does the multiplication backward work?* and *Did we get all the common factors out?* Applying the distributive law, we see that the multiplication backward does work. Inspecting the three terms inside the parentheses, we see that they do not share any divisors. This is especially easy, given that the second term is -1. Thus our solution is correct.

Sometimes we will need to factor out -1 from an expression. This step is usually needed when the coefficient of the highest degree term is -1.

Example 3. Factor out -1 from $8x^5 - x^6 + 3x - 2$.

Solution: It is always a good idea to rearrange the terms by degree. Then we write $-1\left(\qquad \right)$. Inside the parentheses, we write the opposite of our expression, i.e. change all signs.

$$8x^5 - x^6 + 3x - 2 = -x^6 + 8x^5 + 3x - 2 = \boxed{-1\left(x^6 - 8x^5 - 3x + 2\right)}$$

We often omit the 1 and write only $-\left(x^6 - 8x^5 - 3x + 2\right)$.

Sometimes the greatest common factor is more complicated.

Example 4. Factor out the GCF from $12a^3(a-2) - 6a^2(a-2) + 24(a-2)$.

Solution: In this case, $a-2$ is part of the GCF. We factor it out:

$$12a^3(a-2) - 6a^2(a-2) + 24(a-2) = (a-2)\left(12a^3 - 6a^2 + 24\right)$$

If we look at the expression in the second pair of parentheses, we see that there is a common factor of 6. Thus the final answer is

$$(a-2)\,6\left(2a^3 - a^2 + 4\right) = \boxed{6(a-2)\left(2a^3 - a^2 + 4\right)}$$

Factoring out the GCF must always be the first step in factoring. In case of the next example, this is all we need.

Example 5. Solve the equation $x^2 = 6x$

Solution: We realize that this is a quadratic equation. Therefore, we need to reduce one side to zero, factor, and apply the zero product rule. The number multiplying the variables in the highest degree term is called **the leading coefficient**. When reducing one side to zero, we should try to avoid creating negative leading coefficients. In this case, we should subtract $6x$ from both sides.

$$\begin{aligned} x^2 &= 6x && \text{subtract } 6x \\ x^2 - 6x &= 0 && \text{factor out the GCF} \\ x(x-6) &= 0 \end{aligned}$$

We apply the zero product rule to the two factors:

$$x = 0 \quad \text{or} \quad x - 6 = 0$$
$$x = 6$$

Therefore, there are two solutions, $\boxed{0 \text{ and } 6}$. We check: if $x = 0$, then both sides are zero. If $x = 6$, then both sides are 36. Thus our solution is correct.

Example 6. Find all numbers with the following property. The number raised to the third power is five times the number we get if we double the number and then square the result.

Solution: We label this number by x. Then the number raised to the third power is x^3. If we double the number, we get $2x$. We write the equation comparing the square of $2x$ and x^3.

$$\begin{aligned} 5\left((2x)^2\right) &= x^3 \\ 5\left(4x^2\right) &= x^3 \\ 20x^2 &= x^3 && \text{subtract } 20x^2 \\ 0 &= x^3 - 20x^2 && \text{factor out the GCF} \\ 0 &= x^2(x-20) && \text{apply the zero product rule} \end{aligned}$$

$$x = 0 \text{ or } x = 20$$

So there are two such numbers: $\boxed{0 \text{ and } 20}$. We check: 0 clearly works. If the number is 20, it raised to the third power is $20^3 = 8000$. If we double 20, we get 40. The square of 40 is $40^2 = 1600$, and indeed 8000 is five times 1600, thus our solution is correct.

Part 2 – The Difference of Squares Theorem

Consider a sum or a difference such as $x-5$ or $2a+1$. If we change both signs in such an expression, we obtain its opposite. When we change only one of the two signs, we obtain its conjugate.

Definition: Two algebraic expressions are **conjugates** if they both have two terms and are identical except for the sign of one of the terms. For example, $x-5$ and $x+5$ are conjugates of each other. So are $2a-1$ and $2a+1$.

Conjugates are very useful in algebra for all kinds of reasons. Perhaps their most important advantage is their behavior when multiplied. Consider a few examples.

$$\begin{aligned}(x-5)(x+5) &= x^2-5x+5x-25 = x^2-25\\ (2a+1)(2a-1) &= 4a^2-2a+2a+1 = 4a^2-1\end{aligned}$$

Because of the identical terms and alternating signs, O and I from FOIL completely cancel out each other, and we are left with only two terms. In general, when we multiply conjugates $A+B$ and $A-B$, where A could be any number or expression, $(A+B)(A-B) = A^2-AB+AB-B^2 = A^2-B^2$ and therefore

$$(A+B)(A-B) = A^2-B^2$$

This statement is very clear, easy to understand, and completely mechanical. But it becomes much less clear, almost mysterious when we apply the equality backwards.

Theorem: (The Difference of Squares Theorem) If A and B are any number or expression, the difference of their squares can always be factored into a pair of conjugates:

$$A^2-B^2 = (A+B)(A-B)$$

Example 7. Completely factor each of the following.

a) x^2-25 b) $18a^2x-8b^2x$ c) $(5m+3n-1)^2-(-2m-n+5)^2$

Solution: a) We realize that we are looking at a difference between two squares. By the difference of squares theorem, such an expression can always be factored into a pair of conjugates.

$$\begin{aligned}x^2-25 &= x^2-5^2\\ &= (x+5)(x-5)\end{aligned}$$

So our answer is $\boxed{(x+5)(x-5)}$.

b) The terms in $18a^2x-8b^2x$ are not all squares. This is because there is a GCF that needs to be factored out before we could apply the difference of squares theorem.

$$\begin{aligned}18a^2x-8b^2x &= 2x\left(9a^2-4b^2\right) && \text{realize the setup for the difference of squares theorem}\\ &= 2x\left((3a)^2-(2b)^2\right) && \text{factor via the theorem}\\ &= 2x(3a+2b)(3a-2b)\end{aligned}$$

To completely factor an expression, we often use several techniques. **The GCF must always be the first one** because, as this example shows, sometimes the GCF is an obstacle to applying other factoring techniques.

c) This example is here to remind students how mechanical this theorem really is. In the statement $A^2 - B^2 = (A+B)(A-B)$, A and B could be any algebraic expressions, not just a number. For example, $A = 5m + 3n - 1$ and $B = -2m - n + 5$. If we state the difference of squares theorem with these expression, then $A^2 - B^2 = (A+B)(A-B)$ becomes

$$\begin{aligned}(5m+3n-1)^2 - (-2m-n+5)^2 &= \\ &= [(5m+3n-1)+(-2m-n+5)][(5m+3n-1)-(-2m-n+5)] \\ &= (5m+3n-1-2m-n+5)(5m+3n-1+2m+n-5) \quad \text{combine like terms} \\ &= (3m+2n+4)(7m+4m-6)\end{aligned}$$

This problem would be quite difficult to solve using other methods.

Example 8. Completely factor each of the following.

a) $x^2 + 9$ b) $x^{18}y - 25x^2y^5$ c) $80x^4 - 5$

Solution: a) The expression $x^2 + 9$ is not the difference of two squares, rather, it is their sum. **The sum of two squares can not be factored.** Therefore, the final alswer is $\boxed{x^2 + 9}$.

b) We factor out the GCF first. The GCF in $x^{18}y - 25x^2y^5$ is x^2y.

$x^{18}y - 25x^2y^5 = x^2y\left(x^{16} - 25y^2\right)$

We might be tempted to think that the square root of x^{16} is x^4. This is not true, however. Recall that $(a^n)^m = a^{nm}$ and so $\left(a^8\right)^2 = a^{16}$. The square root of x^{16} is x^8. We realize the diiference of squares and then factor it into a pair of conjugates.

$x^2y\left(x^{16} - 25y^2\right) = x^2y\left(\left(x^8\right)^2 - (5y)^2\right) = \boxed{x^2y\left(x^8 + 5y\right)\left(x^8 - 5y\right)}$

c) We factor out the GCF first. Then, if we see that difference of squares theorem, we apply it.

$80x^4 - 5 = 5\left(16x^4 - 1\right) = 5\left(\left(4x^2\right)^2 - 1^2\right) = 5\left(4x^2 + 1\right)\left(4x^2 - 1\right)$

We are not done yet. The expression $4x^2 + 1$ is a sum of two squares, therefore it can not be factored further. But $4x^2 - 1$ is a difference of two squares, and can be therefore factored into a pair of conjugates.

$5\left(4x^2 + 1\right)\left(4x^2 - 1\right) = 5\left(4x^2 + 1\right)\left((2x)^2 - 1^2\right) = \boxed{5\left(4x^2 + 1\right)(2x + 1)(2x - 1)}$

This happens when we have the difference of two quantities raised to the fourth power. The difference of squares theorem can be applied twice.

Example 9. Solve the equation $3x^3 = 12x$.

Solution: If the equation is of a degree higher than one, we need to apply the zero product rule. We reduce one side to zero, and factor the other side.

$$\begin{aligned}3x^3 &= 12x & & \\ 3x^3 - 12x &= 0 & & \text{factor out the GCF} \\ 3x\left(x^2 - 4\right) &= 0 & & \text{realize the difference of squares} \\ 3x\left(x^2 - 2^2\right) &= 0 & & \text{apply it} \\ 3x(x+2)(x-2) &= 0 & &\end{aligned}$$

We apply the zero product rule. We can treat $3x$ as two different factors or just one factor.

$$\begin{aligned}3x = 0 \quad &\text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 2 = 0 \\ x = 0 \quad &\text{or} \quad \quad x = -2 \quad \text{or} \quad x = 2\end{aligned}$$

We check our solutions.

If $x=0$, then LHS $=3\cdot 0^3=0$ and RHS $=12\cdot 0=0$ $\checkmark$.

If $x=-2$, then LHS $=3\cdot(-2)^3=3(-8)=-24$ and RHS $=12(-2)=-24$ $\checkmark$.

If $x=2$, then LHS $=3\cdot 2^3=3\cdot 8=24$ and RHS $=12\cdot 2=24$ $\checkmark$.

Thus, our solution, $\boxed{0, -2, \text{ and } 2}$ is correct.

Example 10. If we raise a number to the third power, we get nine times the number. Find all numbers with this property.

Solution: We label the unknown number by x. The equation is then $x^3=9x$. We solve this equation.

$$\begin{aligned}
x^3 &= 9x && \text{subtract } 9x \text{ to reduce one side to zero}\\
x^3-9x &= 0 && \text{factor out the GCF}\\
x\left(x^2-9\right) &= 0 && \text{realize the difference of two squares}\\
x\left(x^2-3^2\right) &= 0 && \text{and then apply it}\\
x(x+3)(x-3) &= 0 && \text{apply the zero product rule}
\end{aligned}$$

$$x=0 \quad \text{or} \quad x+3=0 \quad \text{or} \quad x-3=0$$

$$x=0 \quad \text{or} \quad x=-3 \quad \text{or} \quad x=3$$

We check against the conditions stated in the problem. Clearly, 0^3 is nine times 0. Similarly, $3^3=27$ is nine times 3, and $(-3)^3=-27$ is nine times -3. Therefore, our solution, $\boxed{0, 3, \text{ and } -3}$ is correct.

The difference of squares theorem also has some practical applications to arithmetic. If we have to compute the difference of two large squares that have an easily computable sum or difference, we can apply the theorem to cut down on computation.

Example 11. Compute each of the following without using a calculator.

a) 52^2-48^2 b) 100^2-99^2

Solution: a) Notice that the sum of 52 and 48 is 100. (Before the addition, take away 2 from 52 and add it to 48) and their difference is 4. Let us apply the difference of squares theorem.

$52^2-48^2=(52+48)(52-48)=100\cdot 4=\boxed{400}$.

b) The difference between 100 and 99 is 1, therefore 100^2-99^2 will be the same as the sum of 100 and 99.

$100^2-99^2=(100+99)(100-99)=199\cdot 1=\boxed{199}$.

Discussion:
While x^2-9 can be factored via the difference of squares theorem, x^2+9 can not be factored. How are these two facts related to the equations $x^2=9$ and $x^2=-9$?

Sample Problems

1. Completely factor each of the following.

a) $3x-12$ c) $3a^2-12$ e) x^2-1 g) $-49+x^6$ i) $2p^4-162$

b) x^2-25y^2 d) $3a^2-12a$ f) x^2+1 h) $3a^3-27ab^2$ j) $20x+5x^3$

2. Solve each of the following equations. Make sure to check your solution.

a) $(x-2)(x+3)(2x+1)=0$ c) $x^2=9$ e) $8x^3=50x^2$

b) $m(m+7)=0$ d) $x^2=9x$ f) $8p^3=50p$

3. Word Problems

a) Find all numbers that satisfy the following condition: if we square the number, we get back the same number.

b) Find all numbers that satisfy the following condition: if we raise the number to the third power, the result is four times the original number.

Practice Problems

1. Factor out the greatest common factor from each of the following.

a) $10a^2b^2-15ab^3+25a^2b^3c$ c) $a^2-a^3+a^4$ e) $x^5-2x^4+4x^3$

b) $6x^3-3x^2-15x^4$ d) $6a^2b+12a^3b-30a^3b^2$ f) $3xy(a-3)+8t(a-3)-200x^5(a-3)$

2. Factor out -1 from each of the following.

a) x^3-x^5+2 b) $-x^2+3x-1$ c) $-x^2+3x-5$

3. Factor each of the following via the difference of squares theorem.

a) x^2-49 b) $9a^2-25$ c) x^2-1 d) y^6-100

4. Completely factor each of the following.

a) $5a^2-45$ e) x^3-x i) $a^2-(x-1)^2$ m) $-2x^4+162$

b) $2m^4-2n^4$ f) $5x^3y^4-80x^3$ j) $-16+a^4$ n) $5a^3b^2-15ab$

c) $2x^4-8x^2$ g) $a^2(x-1)-9(x-1)$ k) $600ab^2-6ab^4$

d) $3a-12ab^2$ h) $18a^2x^2-50x^2$ l) $36x^2y^3+4x^4y^3$

5. Solve each of the following equations. Make sure to check your solutions.

a) $(w+5)(w-1)=0$ c) $2(x-2)(x+3)=0$ e) $x^2+6x=0$ g) $3x^3=75x$

b) $x(x-2)(x+3)=0$ d) $x^2=4$ f) $3x^3=75x^2$ h) $45a^4=20a^2$

6. Find all numbers satisfying the given conditions.

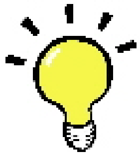
   a) The cube of the number is three times as large as the square of twice the number.

   b) The cube of the number is five times as large as the opposite of the square of the number.

   c) The cube of a number is the same as the four times the number.

7. Use the difference of squares theorem to compute the following without a calculator.

   a) $51^2 - 49^2$ b) $2001^2 - 2000^2$ c) $120^2 - 20^2$ d) $28^2 - 22^2$



Answers

Discussion:

The equation $x^2 = 9$ has two solutions, $x = 3$ and -3. We can solve this equation by factoring:

$$\begin{array}{cc} x^2 = 9 & x^2 = -9 \\ x^2 - 9 = 0 & x^2 + 9 = 0 \\ (x+3)(x-3) = 0 & (\ \ ?\ \ )(\ \ ?\ \ ) = 0 \\ x_1 = -3 \text{ and } x_2 = 3 & \end{array}$$

If $x^2 + 9$ could be factored, then both linear factors would yield for a solution. But the equation $x^2 = -9$ has no solution because the square of no real number is negative. Therefore, $x^2 + 9$ cannot be factored.

Sample Problems

1. a) $3(x-4)$ b) $(x+5y)(x-5y)$ c) $3(a+2)(a-2)$ d) $3a(a-4)$ e) $(x+1)(x-1)$ f) x^2+1
   g) $\left(x^3+7\right)\left(x^3-7\right)$ h) $3a(a+3b)(a-3b)$ i) $2\left(p^2+9\right)(p+3)(p-3)$ j) $5x\left(x^2+4\right)$
2. a) $2, -3$, and $-\dfrac{1}{2}$ b) 0 and -7 c) -3 and 3 d) 0 and 9 e) 0 and $\dfrac{25}{4}$ f) $-\dfrac{5}{2}$, 0, and $\dfrac{5}{2}$
3. a) $0, 1$ b) $0, 2, -2$

Practice Problems

1. a) $5ab^2(2a-3b+5abc)$ b) $3x^2\left(2x-5x^2-1\right)$ c) $a^2\left(a^2-a+1\right)$ d) $6a^2b(2a-5ab+1)$
   e) $x^3\left(x^2-2x+4\right)$ f) $(a-3)\left(3xy+8t-200x^5\right)$
2. a) $-\left(-x^3+x^5-2\right)$ b) $-\left(x^2-3x+1\right)$ c) $-\left(x^2-3x+5\right)$
3. a) $(x+7)(x-7)$ b) $(3a+5)(3a-5)$ c) $(x+1)(x-1)$ d) $\left(y^3+10\right)\left(y^3-10\right)$
4. a) $5(a+3)(a-3)$ b) $-2(n-m)(m+n)\left(m^2+n^2\right)$ c) $2x^2(x+2)(x-2)$ d) $-3a(2b+1)(2b-1)$
   e) $x(x+1)(x-1)$ f) $5x^3(y-2)(y+2)\left(y^2+4\right)$ g) $(a-3)(a+3)(x-1)$ h) $2x^2(3a-5)(3a+5)$
   i) $(a+x-1)(a-x+1)$ j) $(a-2)(a+2)\left(a^2+4\right)$ k) $-6ab^2(b-10)(b+10)$ l) $4x^2y^3\left(x^2+9\right)$
   m) $-2\left(x^2+9\right)(x+3)(x-3)$ n) $5ab\left(a^2b-3\right)$
5. a) $-5, 1$ b) $0, 2, -3$ c) $2, -3$ d) $2, -2$ e) $0, -6$ f) $0, 25$ g) $-5, 0, 5$ h) $-\dfrac{2}{3}, 0, \dfrac{2}{3}$
6. a) $0, 12$ b) $0, -5$ c) $-2, 0, 2$ 7. a) 200 b) 4001 c) $14\,000$ d) 300

Sample Problems Solutions

1. Completely factor each of the following.

a) $3x - 12$

Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 3.

$$3x - 12 = \boxed{3(x-4)}$$

What is in the parentheses, $x - 4$ can not be further factored. We can easily check our work by multiplication.

b) $x^2 - 25y^2$

Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. However, $x^2 - 25y^2$ can be factored via the difference of squares theorem.

$$x^2 - 25y^2 = x^2 - (5y)^2 = (x+5y)(x-5y)$$

The expressions in neither parentheses can be further factored and so we sre done. We check our work by multiplication:

$$(x+5y)(x-5y) = x^2 - 5xy + 5xy - 25 = x^2 - 25y^2$$

and so our answer, $\boxed{(x+5y)(x-5y)}$ is correct.

c) $3a^2 - 12$

Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 3.

$$3a^2 - 12 = 3\left(a^2 - 4\right)$$

What is in the parentheses, $a^2 - 4$ can be further factored via the difference of squares theorem.

$$3\left(a^2 - 4\right) = 3\left(a^2 - 2^2\right) = 3(a+2)(a-2)$$

The expressions in neither parentheses can be further factored and so we sre done. We check our work by multiplication:

$$3(a+2)(a-2) = 3\left(a^2 - 2a + 2a - 4\right) = 3\left(a^2 - 4\right) = 3a^2 - 12$$

and so our answer, $\boxed{3(a+2)(a-2)}$ is correct.

d) $3a^2 - 12a$

Solution: This problem, together with the previous one, illustrates that two problems might look very similar, those small differences are quite significant when it comes to the solution and to the techniques we need to use to solve them. We start with the greatest common factor (or GCF). In this case, the GCF is $3a$.

$$3a^2 - 12a = 3a(a-4)$$

What is in the parentheses, $a - 4$ can not be further factored and so we are done. We can easily check our work by multiplication:

$$3a(a-4) = 3a^2 - 12a$$

and so our answer, $\boxed{3a(a-4)}$ is correct.

e) $x^2 - 1$

Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. However, $x^2 - 1$ can be factored via the difference of squares theorem.

$$x^2 - 1 = x^2 - 1^2 = (x+1)(x-1)$$

The expressions in neither parentheses can not be further factored and so we sre done. We check our work by multiplication:

$$(x+1)(x-1) = x^2 - x + x - 1 = x^2 - 1$$

and so our answer, $\boxed{(x+1)(x-1)}$ is correct. This is probably the most commonly occurring difference of two squares.

f) $x^2 + 1$

Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. In addition, $x^2 + 1$ can NOT be factored via the difference of squares theorem. **The sum of two squares can never be factored.** So, there is nothing that can be done here, and the final answer is $\boxed{x^2 + 1}$.

g) $-49 + x^6$

Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. Before we proceed any further, we rearrange the terms so that the difference of squares becomes easier to observe.

$$-49 + x^6 = x^6 - 49$$

This factors via the difference of sqaures theorem. It is x^3 that we need to square to obtain x^6.

$$x^6 - 49 = \left(x^3\right)^2 - 7^2 = \left(x^3 + 7\right)\left(x^3 - 7\right)$$

What is in both parentheses, $x^3 + 7$ and $x^3 - 7$ can not be further factored and so we are done. We can easily check our work by multiplication:

$$\left(x^3 + 7\right)\left(x^3 - 7\right) = x^6 - 7x^3 + 7x^3 - 49 = x^6 - 49$$

and so our answer, $\boxed{\left(x^3 + 7\right)\left(x^3 - 7\right)}$ is correct.

h) $3a^3 - 27ab^2$

Solution: We start with the greatest common factor (or GCF).

$$\begin{aligned}
3a^3 - 27ab^2 &= \text{factor out GCF} \\
3a\left(a^2 - 9b^2\right) &= \text{re-write } 9b^2 \text{ as } (3b)^2 \\
3a\left(a^2 - (3b)^2\right) &= \text{factor via the difference of squares theorem} \\
&= 3a(a+3b)(a-3b)
\end{aligned}$$

We check by multiplication:

$$3a(a+3b)(a-3b) = 3a\left(a^2 - 3ab + 3ab - 9b^2\right) = 3a\left(a^2 - 9b^2\right) = 3a^3 - 27ab^2$$

Thus our solution, $\boxed{3a(a+3b)(a-3b)}$ is correct.

i) $2p^4 - 162$

Solution: We start with the greatest common factor (or GCF).

$$\begin{aligned}
2p^4 - 162 &= \quad \text{factor out GCF} \\
2\left(p^4 - 81\right) &= \quad \text{re-write both quantities as squares} \\
2\left(\left(p^2\right)^2 - 9^2\right) &= \quad \text{factor via the difference of squares theorem} \\
2\left(p^2+9\right)\left(p^2-9\right) &= \quad \text{second factor will factor again} \\
2\left(p^2+9\right)\left(p^2-3^2\right) &= \quad \text{factor via the difference of squares theorem} \\
&= 2\left(p^2+9\right)(p+3)(p-3)
\end{aligned}$$

We check by multiplication:

$$\begin{aligned}
2\left(p^2+9\right)\underbrace{(p+3)(p-3)}_{\text{FOIL}} &= 2\left(p^2+9\right)\left(p^2-3p+3p-9\right) = 2\underbrace{\left(p^2+9\right)\left(p^2-9\right)}_{\text{FOIL}} \\
&= 2\left(p^4-9p^2+9p^2-81\right) = 2\left(p^4-81\right) = 2p^4-162
\end{aligned}$$

Thus our solution, $\boxed{2\left(p^2+9\right)(p+3)(p-3)}$ is correct.

j) $20x + 5x^3$

Solution: We rearrange the terms by degree first and then factor out the GCF.

$$20x + 5x^3 = 5x^3 + 20x = 5x\left(x^2+4\right)$$

Since the sum of squares does not factor, the final answer is $\boxed{5x\left(x^2+4\right)}$. We can easily check the result by mulitplication.

2. Solve each of the following equations. Make sure to check your solution.

a) $(x-2)(x+3)(2x+1) = 0$

Solution: Since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule. Most of these were already done for us as the right-hand side is zero and the left-hand side is completely factored. All we need to do is apply the zero product rule. **A product can only be zero if one of its factors is zero.** $(x-2)(x+3)(2x+1) = 0$ means that either $x-2=0$ or $x+3=0$ or $2x+1=0$. We solve these linear equations separately:

$$\begin{array}{ccccc}
x-2=0 & \text{or} & x+3=0 & \text{or} & 2x+1=0 \\
x=2 & & x=-3 & & 2x=-1 \\
 & & & & x=-\dfrac{1}{2}
\end{array}$$

We check all three solutions. If $x=2$, then LHS $= (2-2)(2+3)(2(2)+1) = 0 \cdot 5 \cdot 5 = 0 =$ RHS ✓

If $x=-3$, then LHS $= (-3-2)(-3+3)(2(-3)+1) = -5 \cdot 0 \cdot (-5) = 0 =$ RHS ✓

and if $x = -\dfrac{1}{2}$, then

$$\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}+3\right)\left(2\left(-\frac{1}{2}\right)+1\right) = -\frac{3}{2}\cdot\frac{5}{2}\cdot 0 = 0$$

and so all three numbers, $\boxed{2, -3, \text{ and } -\dfrac{1}{2}}$ are correct.

b) $m(m+7)=0$

Solution: We will apply the zero product rule. **A product can only be zero if one of its factors is zero.** $m(m+7)=0$ means that either $m=0$. We solve these linear equations separately and obtain $m=0$ and $m=-7$. We check: If $m=0$, then

$$0(0+7)=0\cdot 7=0$$

and if $m=-7$, then

$$-7(-7+7)=-7\cdot 0$$

and so both numbers, $\boxed{0 \text{ and } -7}$ are correct.

c) $x^2=9$

Solution: Since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} x^2 &= 9 && \text{subtract } 9 \\ x^2-9 &= 0 && \text{factor via the difference of squares theorem} \\ x^2-3^2 &= 0 \\ (x+3)(x-3) &= 0 \end{aligned}$$

A product can only be zero if one of its factors is zero. $(x+3)(x-3)=0$ means that either $x-3=0$ or $x+3=0$. We solve these linear equations separately and obtain $\boxed{3 \text{ and } -3}$. We check: $3^2=9$ and $(-3)^2=9$.

Note: one could ask why the four steps if we could just conclude from $x^2=9$ that then $x=\pm 3$. This shortcut (called the square root property) is perfectly fine, as long as we remember that there are two numbers whose square is 9: 3 and -3. It is a common and serious error to go from $x^2=9$ to $x=3$. One advantage of the difference of squares theorem that it will not allow for this mistake.

d) $x^2=9x$

Solution: Since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} x^2 &= 9x && \text{subtract } 9x \\ x^2-9x &= 0 && \text{factor out the GCF} \\ x(x-9) &= 0 \end{aligned}$$

A product can only be zero if one of its factors is zero. $x(x-9)=0$ means that either $x=0$ or $x-9=0$. We solve these linear equations separately and obtain $\boxed{0 \text{ and } 9}$. We check: $0^2=9\cdot 0$ and $9^2=9\cdot 9$ and so our solution is correct.

e) $8x^3 = 50x^2$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} 8x^3 &= 50x^2 && \text{subtract } 50x^2 \\ 8x^3 - 50x^2 &= 0 && \text{the GCF is } 2x^2 \\ 2x^2(4x-25) &= 0 \end{aligned}$$

We now apply the zero product rule. If this product is zero, then either $2x^2 = 0$ or $4x - 25 = 0$. We solve these equations for x.

$$\begin{aligned} 2x^2 &= 0 && \text{or} && 4x - 25 = 0 \\ 2 \cdot x \cdot x &= 0 && \text{or} && 4x = 25 \\ x &= 0 && \text{or} && x = \frac{25}{4} \end{aligned}$$

We check both solutions. If $x = 0$, then LHS $= 8 \cdot 0^3 = 8 \cdot 0 = 0$ and RHS $= 50 \cdot 0^2 = 50 \cdot 0 = 0$ ✓
If $x = \frac{25}{4}$, then

$$\text{LHS} = 8\left(\frac{25}{4}\right)^3 = \frac{8}{1} \cdot \frac{15\,625}{64} = \frac{15\,625}{8} \quad \text{and} \quad \text{RHS} = 50\left(\frac{25}{4}\right)^2 = \frac{50}{1} \cdot \frac{625}{16} = \frac{15\,625}{8} \checkmark$$

Thus both solutions, $\boxed{0 \text{ and } \frac{25}{4}}$ are correct.

f) $8p^3 = 50p$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} 8p^3 &= 50p && \text{subtract } 50p \\ 8p^3 - 50p &= 0 && \text{the GCF is } 2p \\ 2p\left(4p^2 - 25\right) &= 0 \\ 2p\left((2p)^2 - 5^2\right) &= 0 && \text{factor via difference of squares theorem} \\ 2p(2p+5)(2p-5) &= 0 \end{aligned}$$

We now apply the special zero property. If this product is zero, then either $2p = 0$ or $2p + 5 = 0$ or $2p - 5 = 0$. We solve these equations for p.

$$\begin{aligned} 2p + 5 &= 0 && \text{or} && 2p - 5 = 0 && \text{or} && 2p = 0 \\ 2p &= -5 && \text{or} && 2p = 5 && \text{or} && p = 0 \\ p &= -\frac{5}{2} && \text{or} && p = \frac{5}{2} \end{aligned}$$

We check all three solutions. If $p = -\frac{5}{2}$, then

$$\text{LHS} = 8\left(-\frac{5}{2}\right)^3 = \frac{8}{1} \cdot \frac{-125}{8} = -125 \quad \text{and} \quad \text{RHS} = 50\left(-\frac{5}{2}\right) = \frac{50}{1} \cdot \frac{-5}{2} = \frac{-250}{2} = -125 \checkmark$$

And if $p=\frac{5}{2}$, then LHS $=8\left(\frac{5}{2}\right)^3=\frac{8}{1}\cdot\frac{125}{8}=125$ and RHS $=50\left(\frac{5}{2}\right)=\frac{50}{1}\cdot\frac{5}{2}=\frac{250}{2}=125$ ✓

And if $p=0$, then LHS $=8\cdot 0^3=8\cdot 0=0$ and RHS $=50\cdot 0=0$ ✓

Thus all three solutions, $\boxed{-\frac{5}{2},\ 0,\ \text{and}\ \frac{5}{2}}$ are correct.

3. Word Problems

a) Find all numbers that satisfy the following condition: if we square the number, we get back the same number.

Solution: Let us denote the number by x. The equation is

$$\begin{aligned} x^2 &= x && \text{reduce one side to zero} \\ x^2-x &= 0 && \text{factor} \\ x\,(x-1) &= 0 && \text{apply the zero property} \end{aligned}$$

$$\begin{array}{ccccc} x = 0 & \text{or} & x-1=0 \\ x = 0 & \text{or} & x=1 \end{array}$$

Thus there are two numbers, 0 and 1, satisfying the property. We check: $0^2=0$ and $1^2=1$.
Thus our answer is: $\boxed{0 \text{ and } 1}$.

b) Find all numbers that satisfy the following condition: if we raise the number to the third power, the result is four times the original number.

Solution: Let us denote the number by x. The equation is

$$\begin{aligned} x^3 &= 4x && \text{reduce one side to zero} \\ x^3-4x &= 0 && \text{factor out the GCF} \\ x\left(x^2-4\right) &= 0 && \text{factor via the difference of squares theorem} \\ x\,(x+2)\,(x-2) &= 0 && \text{apply the zero property} \end{aligned}$$

$$\begin{array}{ccccc} x = 0 & \text{or} & x+2=0 & \text{or} & x-2=0 \\ x = 0 & \text{or} & x=-2 & \text{or} & x=2 \end{array}$$

Thus there are three numbers, 0, 2 and -2, satisfying the property. We check: $0^3=4\cdot 0,\ 2^3=4\cdot 2,$ and $-2^3=4\,(-2)$.
Thus our answer is: $\boxed{0,\ 2,\ \text{and}\ -2.}$

