Factoring out the GCF and the Difference of Squares Theorem

Sample Problems

1. Completely factor each of the following.

   a) $3x - 12$ c) $3a^2 - 12$ e) $x^2 - 1$ g) $-49 + x^6$ i) $2p^4 - 162$

   b) $x^2 - 25y^2$ d) $3a^2 - 12a$ f) $x^2 + 1$ h) $3a^3 - 27ab^2$ j) $20x + 5x^3$

2. Solve each of the following equations. Make sure to check your solution.

   a) $(x-2)(x+3)(2x+1) = 0$ c) $x^2 = 9$ e) $8x^3 = 50x^2$

   b) $m(m+7) = 0$ d) $x^2 = 9x$ f) $8p^3 = 50p$

3. Word Problems

   a) Find all numbers that satisfy the following condition: if we square the number, we get back the same number.

   b) Find all numbers that satisfy the following condition: if we raise the number to the third power, the result is four times the original number.

Practice Problems

1. Factor out the greatest common factor from each of the following.

   a) $10a^2b^2 - 15ab^3 + 25a^2b^3c$ c) $a^2 - a^3 + a^4$ e) $x^5 - 2x^4 + 4x^3$

   b) $6x^3 - 3x^2 - 15x^4$ d) $6a^2b + 12a^3b - 30a^3b^2$ f) $3xy(a-3) + 8t(a-3) - 200x^5(a-3)$

2. Factor out -1 from each of the following.

   a) $x^3 - x^5 + 2$ b) $-x^2 + 3x - 1$ c) $-x^2 + 3x - 5$

3. Factor each of the following via the difference of squares theorem.

   a) $x^2 - 49$ b) $9a^2 - 25$ c) $x^2 - 1$ d) $y^6 - 100$

4. Completely factor each of the following.

   a) $5a^2 - 45$ e) $x^3 - x$ i) $a^2 - (x-1)^2$ m) $-2x^4 + 162$

   b) $2m^4 - 2n^4$ f) $5x^3y^4 - 80x^3$ j) $-16 + a^4$ n) $5a^3b^2 - 15ab$

   c) $2x^4 - 8x^2$ g) $a^2(x-1) - 9(x-1)$ k) $600ab^2 - 6ab^4$

   d) $3a - 12ab^2$ h) $18a^2x^2 - 50x^2$ l) $36x^2y^3 + 4x^4y^3$

5. Solve each of the following equations. Make sure to check your solutions.

   a) $(w+5)(w-1) = 0$ c) $2(x-2)(x+3) = 0$ e) $x^2 + 6x = 0$ g) $3x^3 = 75x$

   b) $x(x-2)(x+3) = 0$ d) $x^2 = 4$ f) $3x^3 = 75x^2$ h) $45a^4 = 20a^2$

Answers

Sample Problems

1. a) $3(x-4)$ b) $(x+5y)(x-5y)$ c) $3(a+2)(a-2)$ d) $3a(a-4)$ e) $(x+1)(x-1)$ f) x^2+1
   g) $\left(x^3+7\right)\left(x^3-7\right)$ h) $3a(a+3b)(a-3b)$ i) $2\left(p^2+9\right)(p+3)(p-3)$ j) $5x\left(x^2+4\right)$

2. a) $2, -3$, and $-\frac{1}{2}$ b) 0 and -7 c) -3 and 3 d) 0 and 9 e) 0 and $\frac{25}{4}$ f) $-\frac{5}{2}$, 0, and $\frac{5}{2}$

3. a) $0, 1$ b) $0, 2, -2$

Practice Problems

1. a) $5ab^2(2a-3b+5abc)$ b) $3x^2\left(2x-5x^2-1\right)$ c) $a^2\left(a^2-a+1\right)$ d) $6a^2b(2a-5ab+1)$
   e) $x^3\left(x^2-2x+4\right)$ f) $(a-3)\left(3xy+8t-200x^5\right)$

2. a) $-\left(-x^3+x^5-2\right)$ b) $-\left(x^2-3x+1\right)$ c) $-\left(x^2-3x+5\right)$

3. a) $(x+7)(x-7)$ b) $(3a+5)(3a-5)$ c) $(x+1)(x-1)$ d) $\left(y^3+10\right)\left(y^3-10\right)$

4. a) $5(a+3)(a-3)$ b) $-2(n-m)(m+n)\left(m^2+n^2\right)$ c) $2x^2(x+2)(x-2)$ d) $-3a(2b+1)(2b-1)$
   e) $x(x+1)(x-1)$ f) $5x^3(y-2)(y+2)\left(y^2+4\right)$ g) $(a-3)(a+3)(x-1)$ h) $2x^2(3a-5)(3a+5)$
   i) $(a+x-1)(a-x+1)$ j) $(a-2)(a+2)\left(a^2+4\right)$ k) $-6ab^2(b-10)(b+10)$ l) $4x^2y^3\left(x^2+9\right)$
   m) $-2\left(x^2+9\right)(x+3)(x-3)$ n) $5ab\left(a^2b-3\right)$

5. a) $-5, 1$ b) $0, 2, -3$ c) $2, -3$ d) $2, -2$ e) $0, -6$ f) $0, 25$ g) $-5, 0, 5$ h) $-\frac{2}{3}, 0, \frac{2}{3}$

Sample Problems Solutions

1. Completely factor each of the following.

a) $3x - 12$
Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 3.

$$3x - 12 = \boxed{3\,(x-4)}$$

What is in the parentheses, $x - 4$ can not be further factored. We can easily check our work by multiplication.

b) $x^2 - 25y^2$
Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. However, $x^2 - 25y^2$ can be factored via the difference of squares theorem.

$$x^2 - 25y^2 = x^2 - (5y)^2 = (x+5y)\,(x-5y)$$

The expressions in neither parentheses can be further factored and so we sre done. We check our work by multiplication:

$$(x+5y)\,(x-5y) = x^2 - 5xy + 5xy - 25 = x^2 - 25y^2$$

and so our answer, $\boxed{(x+5y)\,(x-5y)}$ is correct.

c) $3a^2 - 12$
Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 3.

$$3a^2 - 12 = 3\left(a^2 - 4\right)$$

What is in the parentheses, $a^2 - 4$ can be further factored via the difference of squares theorem.

$$3\left(a^2 - 4\right) = 3\left(a^2 - 2^2\right) = 3\,(a+2)\,(a-2)$$

The expressions in neither parentheses can be further factored and so we sre done. We check our work by multiplication:

$$3\,(a+2)\,(a-2) = 3\left(a^2 - 2a + 2a - 4\right) = 3\left(a^2 - 4\right) = 3a^2 - 12$$

and so our answer, $\boxed{3\,(a+2)\,(a-2)}$ is correct.

d) $3a^2 - 12a$
Solution: This problem, together with the previous one, illustrates that two problems might look very similar, those small differences are quite significant when it comes to the solution and to the techniques we need to use to solve them. We start with the greatest common factor (or GCF). In this case, the GCF is $3a$.

$$3a^2 - 12a = 3a\,(a-4)$$

What is in the parentheses, $a - 4$ can not be further factored and so we are done. We can easily check our work by multiplication:

$$3a\,(a-4) = 3a^2 - 12a$$

and so our answer, $\boxed{3a\,(a-4)}$ is correct.

e) $x^2 - 1$
Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. However, $x^2 - 1$ can be factored via the difference of squares theorem.

$$x^2 - 1 = x^2 - 1^2 = (x+1)(x-1)$$

The expressions in neither parentheses can not be further factored and so we sre done. We check our work by multiplication:

$$(x+1)(x-1) = x^2 - x + x - 1 = x^2 - 1$$

and so our answer, $\boxed{(x+1)(x-1)}$ is correct. This is probably the most commonly occurring difference of two squares.

f) $x^2 + 1$
Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. In addition, $x^2 + 1$ can NOT be factored via the difference of squares theorem. **The sum of two squares never factors.** So, there is nothing that can be done here, and the final answer is $\boxed{x^2 + 1}$.

g) $-49 + x^6$
Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. Before we proceed any further, we rearrange the terms so that the difference of squares becomes easier to observe.

$$-49 + x^6 = x^6 - 49$$

This factors via the difference of sqaures theorem. It is x^3 that we need to square to obtain x^6.

$$x^6 - 49 = \left(x^3\right)^2 - 7^2 = \left(x^3 + 7\right)\left(x^3 - 7\right)$$

What is in both parentheses, $x^3 + 7$ and $x^3 - 7$ can not be further factored and so we are done. We can easily check our work by multiplication:

$$\left(x^3 + 7\right)\left(x^3 - 7\right) = x^6 - 7x^3 + 7x^3 - 49 = x^6 - 49$$

and so our answer, $\boxed{\left(x^3 + 7\right)\left(x^3 - 7\right)}$ is correct.

h) $3a^3 - 27ab^2$
Solution: We start with the greatest common factor (or GCF).

$$\begin{aligned}
3a^3 - 27ab^2 &= \quad \text{factor out GCF} \\
3a\left(a^2 - 9b^2\right) &= \quad \text{re-write } 9b^2 \text{ as } (3b)^2 \\
3a\left(a^2 - (3b)^2\right) &= \quad \text{factor via the difference of squares theorem} \\
&= 3a(a+3b)(a-3b)
\end{aligned}$$

We check by multiplication:

$$3a(a+3b)(a-3b) = 3a\left(a^2 - 3ab + 3ab - 9b^2\right) = 3a\left(a^2 - 9b^2\right) = 3a^3 - 27ab^2$$

Thus our solution, $\boxed{3a(a+3b)(a-3b)}$ is correct.

i) $2p^4-162$
Solution: We start with the greatest common factor (or GCF).

$$\begin{aligned}
2p^4-162 &= \quad \text{factor out GCF}\\
2\left(p^4-81\right) &= \quad \text{re-write both quantities as squares}\\
2\left(\left(p^2\right)^2-9^2\right) &= \quad \text{factor via the difference of squares theorem}\\
2\left(p^2+9\right)\left(p^2-9\right) &= \quad \text{second factor will factor again}\\
2\left(p^2+9\right)\left(p^2-3^2\right) &= \quad \text{factor via the difference of squares theorem}\\
&= 2\left(p^2+9\right)(p+3)(p-3)
\end{aligned}$$

We check by multiplication:

$$\begin{aligned}
2\left(p^2+9\right)\underbrace{(p+3)(p-3)}_{\text{FOIL}} &= 2\left(p^2+9\right)\left(p^2-3p+3p-9\right)=2\underbrace{\left(p^2+9\right)\left(p^2-9\right)}_{\text{FOIL}}\\
&= 2\left(p^4-9p^2+9p^2-81\right)=2\left(p^4-81\right)=2p^4-162
\end{aligned}$$

Thus our solution, $\boxed{2\left(p^2+9\right)(p+3)(p-3)}$ is correct.

j) $20x+5x^3$
Solution: We rearrange the terms by degree first and then factor out the GCF.

$$20x+5x^3=5x^3+20x=5x\left(x^2+4\right)$$

Since the sum of squares does not factor, the final answer is $\boxed{5x\left(x^2+4\right)}$. We can easily check the result by mulitplication.

2. Solve each of the following equations. Make sure to check your solution.

a) $(x-2)(x+3)(2x+1)=0$
Solution: Since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule. Most of these were already done for us as the right-hand side is zero and the left-hand side is completely factored. All we need to do is apply the zero product rule. **A product can only be zero if one of its factors is zero.** $(x-2)(x+3)(2x+1)=0$ means that either $x-2=0$ or $x+3=0$ or $2x+1=0$. We solve these linear equations separately:

$$\begin{array}{ccccc}
x-2=0 & \text{or} & x+3=0 & \text{or} & 2x+1=0\\
x=2 & & x=-3 & & 2x=-1\\
 & & & & x=-\dfrac{1}{2}
\end{array}$$

We check all three solutions. If $x=2$, then

$$(2-2)(2+3)(2(2)+1)=0\cdot 5\cdot 5=0$$

If $x=-3$, then

$$(-3-2)(-3+3)(2(-3)+1)=-5\cdot 0\cdot(-5)=0$$

and if $x=-\frac{1}{2}$, then

$$\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}+3\right)\left(2\left(-\frac{1}{2}\right)+1\right)=-\frac{3}{2}\cdot\frac{5}{2}\cdot 0=0$$

and so all three numbers, $\boxed{2,\ -3,\ \text{and}\ -\frac{1}{2}}$ are correct.

b) $m(m+7)=0$
Solution: We will apply the zero product rule. **A product can only be zero if one of its factors is zero.** $m(m+7)=0$ means that either $m=0$. We solve these linear equations separately and obtain $m=0$ and $m=-7$. We check: If $m=0$, then

$$0(0+7)=0\cdot 7=0$$

and if $m=-7$, then

$$-7(-7+7)=-7\cdot 0$$

and so both numbers, $\boxed{0 \text{ and } -7}$ are correct.

c) $x^2=9$
Solution: Since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} x^2 &= 9 && \text{subtract } 9 \\ x^2-9 &= 0 && \text{factor via the difference of squares theorem} \\ x^2-3^2 &= 0 \\ (x+3)(x-3) &= 0 \end{aligned}$$

A product can only be zero if one of its factors is zero. $(x+3)(x-3)=0$ means that either $x-3=0$ or $x+3=0$. We solve these linear equations separately and obtain $\boxed{3 \text{ and } -3}$. We check: $3^2=9$ and $(-3)^2=9$.

Note: one could ask why the four steps if we could just conclude from $x^2=9$ that then $x=\pm 3$. This shortcut (called the square root property) is perfectly fine, as long as we remember that there are two numbers whose square is 9: 3 and -3. It is a common and serious error to go from $x^2=9$ to $x=3$. One advantage of the difference of squares theorem that it will not allow for this mistake.

d) $x^2=9x$
Solution: Since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} x^2 &= 9x && \text{subtract } 9x \\ x^2-9x &= 0 && \text{factor out the GCF} \\ x(x-9) &= 0 \end{aligned}$$

A product can only be zero if one of its factors is zero. $x(x-9)=0$ means that either $x=0$ or $x-9=0$. We solve these linear equations separately and obtain $\boxed{0 \text{ and } 9}$. We check: $0^2=9\cdot 0$ and $9^2=9\cdot 9$ and so our solution is correct.

e) $8x^3=50x^2$
Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} 8x^3 &= 50x^2 && \text{subtract } 50x^2 \\ 8x^3-50x^2 &= 0 && \text{the GCF is } 2x^2 \\ 2x^2(4x-25) &= 0 \end{aligned}$$

We now apply the zero product rule. If this product is zero, then either $2x^2=0$ or $4x-25=0$. We solve these equations for x.

$$\begin{array}{rclcr} 2x^2 &=& 0 & \text{or} & 4x-25=0 \\ 2\cdot x\cdot x &=& 0 & \text{or} & 4x=25 \\ x &=& 0 & \text{or} & x=\dfrac{25}{4} \end{array}$$

We check both solutions. If $x=0$, then

$$\text{LHS} = 8\cdot 0^3 = 8\cdot 0 = 0 \quad \text{and} \quad \text{RHS} = 50\cdot 0^2 = 50\cdot 0 = 0$$

If $x=\dfrac{25}{4}$, then

$$\begin{aligned}\text{LHS} &= 8\left(\frac{25}{4}\right)^3 = \frac{8}{1}\cdot\frac{15\,625}{64} = \frac{15\,625}{8}\\ \text{RHS} &= 50\left(\frac{25}{4}\right)^2 = \frac{50}{1}\cdot\frac{625}{16} = \frac{15\,625}{8}\end{aligned}$$

Thus both solutions, $\boxed{0 \text{ and } \dfrac{25}{4}}$ are correct.

f) $8p^3 = 50p$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned}8p^3 &= 50p && \text{subtract } 50p\\ 8p^3 - 50p &= 0 && \text{the GCF is } 2p\\ 2p\left(4p^2-25\right) &= 0 &&\\ 2p\left((2p)^2 - 5^2\right) &= 0 && \text{factor via difference of squares theorem}\\ 2p\,(2p+5)\,(2p-5) &= 0 &&\end{aligned}$$

We now apply the special zero property. If this product is zero, then either $2p=0$ or $2p+5=0$ or $2p-5=0$. We solve these equations for p.

$$\begin{array}{rclcrcr} 2p+5 &=& 0 & \text{or} & 2p-5=0 & \text{or} & 2p=0\\ 2p &=& -5 & \text{or} & 2p=5 & \text{or} & p=0\\ p &=& -\dfrac{5}{2} & \text{or} & p=\dfrac{5}{2} & & \end{array}$$

We check all three solutions. If $p=-\dfrac{5}{2}$, then

$$\begin{aligned}\text{LHS} &= 8\left(-\frac{5}{2}\right)^3 = \frac{8}{1}\cdot\frac{-125}{8} = -125\\ \text{RHS} &= 50\left(-\frac{5}{2}\right) = \frac{50}{1}\cdot\frac{-5}{2} = \frac{-250}{2} = -125\end{aligned}$$

If $p=\dfrac{5}{2}$, then

$$\begin{aligned}\text{LHS} &= 8\left(\frac{5}{2}\right)^3 = \frac{8}{1}\cdot\frac{125}{8} = 125\\ \text{RHS} &= 50\left(\frac{5}{2}\right) = \frac{50}{1}\cdot\frac{5}{2} = \frac{250}{2} = 125\end{aligned}$$

and if $p=0$, then

$$\text{LHS} = 8\cdot 0^3 = 8\cdot 0 = 0 \quad \text{and} \quad \text{RHS} = 50\cdot 0 = 0$$

Thus all three solutions, $\boxed{-\dfrac{5}{2},\ 0, \text{ and } \dfrac{5}{2}}$ are correct.

3. Word Problems

a) Find all numbers that satisfy the following condition: if we square the number, we get back the same number.
Solution: Let us denote the number by x. The equation is

$$\begin{aligned} x^2 &= x && \text{reduce one side to zero} \\ x^2 - x &= 0 && \text{factor} \\ x(x-1) &= 0 && \text{apply the zero property} \end{aligned}$$

$$\begin{aligned} x &= 0 \qquad \text{or} \qquad x - 1 = 0 \\ x &= 0 \qquad \text{or} \qquad x = 1 \end{aligned}$$

Thus there are two numbers, 0 and 1, satisfying the property. We check: $0^2 = 0$ and $1^2 = 1$. Thus our answer is: $\boxed{\text{0 and 1}}$.

b) Find all numbers that satisfy the following condition: if we raise the number to the third power, the result is four times the original number.
Solution: Let us denote the number by x. The equation is

$$\begin{aligned} x^3 &= 4x && \text{reduce one side to zero} \\ x^3 - 4x &= 0 && \text{factor out the GCF} \\ x\left(x^2 - 4\right) &= 0 && \text{factor via the difference of squares theorem} \\ x(x+2)(x-2) &= 0 && \text{apply the zero property} \end{aligned}$$

$$\begin{aligned} x &= 0 \qquad \text{or} \qquad x + 2 = 0 \qquad \text{or} \qquad x - 2 = 0 \\ x &= 0 \qquad \text{or} \qquad x = -2 \qquad \text{or} \qquad x = 2 \end{aligned}$$

Thus there are three numbers, 0, 2 and -2, satisfying the property. We check: $0^3 = 4 \cdot 0$, $2^3 = 4 \cdot 2$, and $-2^3 = 4(-2)$. Thus our answer is: $\boxed{0,\ 2,\ \text{and}\ -2.}$

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