As we have seen more and more algebraic statements, the solution sets became increasingly more complex. A linear equation usually has a single number solution. In case of linear inequalities, we often have infinitely many solutions. To express those solution sets, we developed interval notation.

Suppose we have an equation in two variables, x and y. The equations $y = 2x - 3$ or $x^2 - y^2 = 5$ or $xy = -2$ are examples for such equations. A solution for such equations is a set of ordered pairs of numbers, (x, y). For example, $(5, 7)$ is short for $x = 5$ and $y = 7$ and this ordered pair is a solution of the equation $y = 2x - 3$. The ordered pair $(3, -2)$ is a solution of $x^2 - y^2 = 5$, and the ordered pair $(2, -1)$ is a solution of $xy = -2$.

Equations in two variables often have infintely many solutions, where that can no longer meaningfully represented on a number line. We step out into two dimensions, and use a coordinate system to depict solution sets. On a coordinate system, each ordered pair (x, y) can be represented as a point.

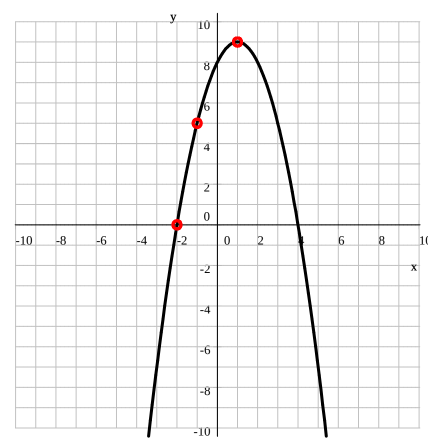
Definition: The **graph** of an equation in x, in y, or both in x and y is the set of all points $P(x, y)$ whose coordinates are solution of the equation.

In short, the graph of an equation is a solution set of an equation in x and y. The shape of graphs depends on the type of equation. Before we started to graph equations, it is useful to know that we can do quite a lot just using the definition of graphs.

Example 1. Consider the graph shown. Three points on the graph are marked. These are $A(-2, 0)$, $B(-1, 5)$, and $C(1, 9)$. Use these points to determine, which of the given equations is the one whose graph is the shape we see.

The possible equations offered are:

$$y = 3x + 6$$
$$(x-4)^2 + (y-5)^2 = 25$$
$$y = -x^2 + 2x + 8$$



Solution: Let us consider first the equation $y = 3x + 6$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A, B, and C. Let's check.

Point $A(-2, 0)$ is on the graph if and only if its coordinates are a solution of $y = 3x + 6$.

Check $y = 3x + 6$ with $x = -2$ and $y = 0$.
The left-hand side is: LHS $= 0$
and the right-hand side is: RHS $= 3(-2) + 6 = 0$. RHS $=$ LHS ✓

Point A is on the graph of $y = 3x + 6$. This does not mean that $y = 3x + 6$ is the right equation. It only means that we didn't rule it out based on point A alone.

Let's see about point $B(-1, 5)$. Is this pont on the graph of $y = 3x + 6$?

Check $y = 3x + 6$ with $x = -1$ and $y = 5$.
LHS $= 5$ and RHS $= 3(-1) + 6 = -3 + 6 = 3$ RHS $\neq$ LHS

At this point, we can conclude that the graph shown is not of the equation of $y = 3x + 6$, because point B is on the graph but its coordinates are not a solution of this equation. So, we can move on to the next equation.

Consider now the equation $(x-4)^2+(y-5)^2=25$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A, B, and C. Let's check.

Point $A\,(-2,0)$ is on the graph if and only if its coordinates are a solution of $(x-4)^2+(y-5)^2=25$.

> Check $(x-4)^2+(y-5)^2=25$ with $x=-2$ and $y=0$.
> LHS $=(-2-4)^2+(0-5)^2=(-6)^2+(-5)^2=36+25=61$
> RHS $=25$. RHS $\neq$ LHS

We can conclude that the graph shown is not of the equation of $(x-4)^2+(y-5)^2=25$, because point A is on the graph but its coordinates are not a solution of this equation. So, we can move on to the next equation.

Consider now the equation $y=-x^2+2x+8$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A, B, and C. Let's check.

Point $A\,(-2,0)$ is on the graph if and only if its coordinates are a solution of $y=-x^2+2x+8$.

> Check $y=-x^2+2x+8$ with $x=-2$ and $y=0$.
> LHS $=0$ and RHS $=-(-2)^2+2(-2)+8=-4-4+8=0$. RHS $=$ LHS ✓

This does not mean that $y=-x^2+2x+8$ is the right equation. It only means that we didn't rule it out based on point A alone. Let's see point B.

Point $B\,(-1,5)$ is on the graph if and only if its coordinates are a solution of $y=-x^2+2x+8$.

> Check $y=-x^2+2x+8$ with $x=-1$ and $y=5$.
> LHS $=5$ and RHS $=-(-1)^2+2(-1)+8=-1-2+8=5$. RHS $=$ LHS ✓

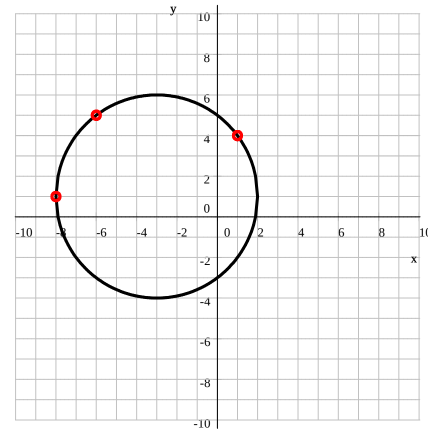
This does not mean that $y=-x^2+2x+8$ is the right equation. It only means that we didn't rule it out based on points A and B. Let's see point C.

Point $C\,(1,9)$ is on the graph if and only if its coordinates are a solution of $y=-x^2+2x+8$.

> Check $y=-x^2+2x+8$ with $x=1$ and $y=9$.
> LHS $=9$ and RHS $=-1^2+2\cdot 1+8=-1+2+8=9$. RHS $=$ LHS ✓

We found that all three points are on the graph of this equation. This still does not mean that $y=-x^2+2x+8$ is the right equation. Given that we were given three equations with the assumption that the correct equation is among them, it can only be this one. So, our answer is that the graph shown is of the equation $y=-x^2+2x+8$. We can find additional nice points on the graph (for example, $(4,0)$ or $(2,8)$) and test them against the equation. Soon we will learn how to graph such shapes.

Example 2. Consider the graph shown. Three points on the graph are marked. These are $A(-8,1)$, $B(-6,5)$, and $C(1,4)$. Use these points to determine, which of the given equations is the one whose graph is the shape we see.



The possible equations offered are:

$3y = x + 11$

$3y + x^2 = -8x + 3$

$(x+3)^2 + (y-1)^2 = 25$

Solution: Let us consider first the equation $3y = x + 11$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A, B, and C. Let's check.

Point $A(-8,1)$ is on the graph if and only if its coordinates are a solution of $3y = x + 11$.

Check $3y = x + 11$ with $x = -8$ and $y = 1$.
The left-hand side is: LHS $= 3 \cdot 1 = 3$
and the right-hand side is: RHS $= -8 + 11 = 3$. RHS $=$ LHS ✓

Point A is on the graph of $3y = x + 11$. This does not mean that $3y = x + 11$ is the right equation. It only means that we didn't rule it out based on point A alone. Let's see point B.

Point $B(-6,5)$ is on the graph if and only if its coordinates are a solution of $3y = x + 11$.

Check $3y = x + 11$ with $x = -6$ and $y = 5$.
LHS $= 3(-6) = -18$ and RHS $= -6 + 11 = 5$. RHS $\neq$ LHS

We can conclude that the graph shown is not of the equation of $3y = x + 11$, because point B is on the graph but its coordinates are not a solution of this equation. So, we can move on to the next equation.

Consider now the equation $3y + x^2 = -8x + 3$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A, B, and C. Let's check.

Point $A(-8,1)$ is on the graph if and only if its coordinates are a solution of $3y + x^2 = -8x + 3$.

Check $3y + x^2 = -8x + 3$ with $x = -8$ and $y = 1$.
LHS $= 3 \cdot 1 + (-8)^2 = 3 + 64 = 67$ and RHS $= -8(-8) + 3 = 64 + 3 = 67$. RHS $=$ LHS ✓

This does not mean that $3y + x^2 = -8x + 3$ is the right equation. It only means that we didn't rule it out based on point A alone. Let's see point B.

Point $B(-6,5)$ is on the graph if and only if its coordinates are a solution of $3y + x^2 = -8x + 3$.

Check $3y + x^2 = -8x + 3$ with $x = -6$ and $y = 5$.
LHS $= 3 \cdot 5 + (-6)^2 = 15 + 36 = 51$ and RHS $= -8(-6) + 3 = 48 + 3 = 51$. RHS $=$ LHS ✓

This does not mean that $3y + x^2 = -8x + 3$ is the right equation. It only means that we didn't rule it out based on points A and B. Let's see point C.

Point $C(1,4)$ is on the graph if and only if its coordinates are a solution of $3y + x^2 = -8x + 3$.

Check $3y + x^2 = -8x + 3$ with $x = 1$ and $y = 4$.
LHS $= 3 \cdot 4 + 4^2 = 12 + 16 = 28$ and RHS $= -8 \cdot 1 + 3 = -5$. RHS $\neq$ LHS

We can conclude that the graph shown is not of the equation of $3y + x^2 = -8x + 3$, because point C is on the graph but its coordinates are not a solution of this equation. So, we can move on to the next equation.

Consider now the equation $(x+3)^2 + (y-1)^2 = 25$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A, B, and C. Let's check.

Point $A(-8,1)$ is on the graph if and only if its coordinates are a solution of $(x+3)^2 + (y-1)^2 = 25$.

Check $(x+3)^2 + (y-1)^2 = 25$ with $x = -8$ and $y = 1$.
LHS $= (-8+3)^2 + (1-1)^2 = (-5)^2 + 0^2 = 25$ and RHS $= 25$. RHS $=$ LHS ✓

This does not mean that $(x+3)^2 + (y-1)^2 = 25$ is the right equation. It only means that we didn't rule it out based on point A alone. Let's see point B.

Point $B(-6,5)$ is on the graph if and only if its coordinates are a solution of $(x+3)^2 + (y-1)^2 = 25$.

Check $(x+3)^2 + (y-1)^2 = 25$ with $x = -6$ and $y = 5$.
LHS $= (-6+3)^2 + (5-1)^2 = (-3)^2 + 4^2 = 9 + 16 = 25$ and RHS $= 25$ RHS $=$ LHS ✓

This does not mean that $(x+3)^2 + (y-1)^2 = 25$ is the right equation. It only means that we didn't rule it out based on points A and B. Let's see point C.

Point $C(1,4)$ is on the graph if and only if its coordinates are a solution of $(x+3)^2 + (y-1)^2 = 25$.

Check $(x+3)^2 + (y-1)^2 = 25$ with $x = 1$ and $y = 4$.
LHS $= (1+3)^2 + (4-1)^2 = 4^2 + 3^2 = 16 + 9 = 25$ and RHS $= 25$ RHS $=$ LHS ✓

We found that all three points are on the graph of this equation. This still does not mean that $(x+3)^2 + (y-1)^2 = 25$ is the right equation. Given that we were given three equations with the assumption that the correct equation is among them, it can only be this one. So, our answer is that the graph shown is of the equation $(x+3)^2 + (y-1)^2 = 25$. We can find additional nice points on the graph (for example, $(2,1)$ or $(-3,-4)$) and test them against the equation.

Practice Problems

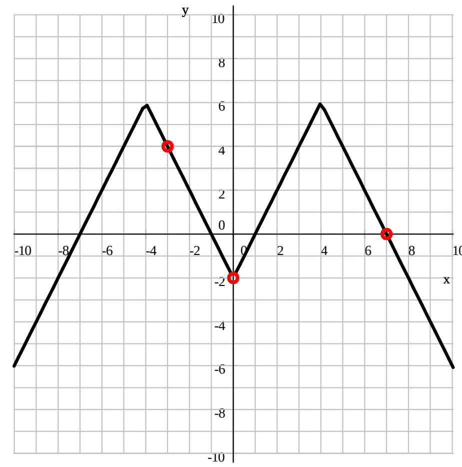
1. Consider the graph shown. Three points on the graph are marked. These are $A(-3,4)$, $B(0,-2)$, and $C(7,0)$. Use these points to determine, which of the given equations is the one whose graph is the shape we see.

   The possible equations offered are:

   $y+2=|2x|$

   $6-y=|8-|2x||$

   $x^2+y^2=1+4(x+y+5)$



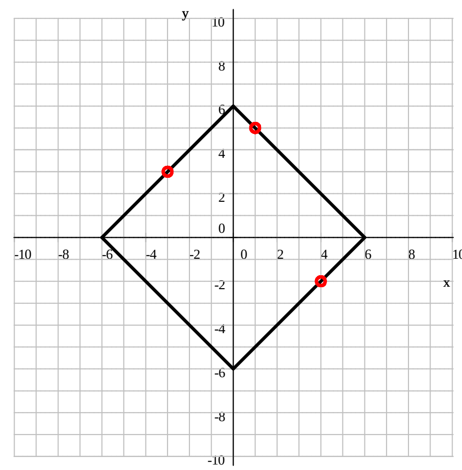
2. Consider the graph shown. Three points on the graph are marked. These are $A(-3,3)$, $B(1,5)$, and $C(4,-2)$. Use these points to determine, which of the given equations is the one whose graph is the shape we see.

   The possible equations offered are:

   $2y=x+9$

   $|x|+|y|=6$

   $y+3=9-|x|$



Answers

1. Consider $y+2=|2x|$

   $A(-3,4)$ is on the graph, $6=6$ ✓
   $B(0,-2)$ is on the graph, $0=0$ ✓
   $C(7,0)$ is not on the graph, $2\neq 14$
   Therefore, $y+2=|2x|$ is not the equation of the graph.

   Consider $6-y=|8-|2x||$

   $A(-3,4)$ is on the graph, $2=2$ ✓
   $B(0,-2)$ is on the graph, $8=8$ ✓
   $C(7,0)$ is not on the graph, $6=6$ ✓
   Therefore, $6-y=|8-|2x||$ is the equation of the graph.

   Consider $x^2+y^2=1+4(x+y+5)$

   $A(-3,4)$ is on the graph, $25=25$ ✓
   $B(0,-2)$ is not on the graph, $4\neq 13$
   $C(7,0)$ is on the graph, $49=49$ ✓
   Therefore, $x^2+y^2=1+4(x+y+5)$ is not the equation of the graph.

2. Consider $2y=x+9$

   $A(-3,3)$ is on the graph, $6=6$ ✓
   $B(1,5)$ is on the graph, $10=10$ ✓
   $C(4,-2)$ is not on the graph, $-4\neq 13$
   Therefore, $2y=x+9$ is not the equation of the graph.

   Consider $|x|+|y|=6$

   $A(-3,3)$ is on the graph, $6=6$ ✓
   $B(1,5)$ is on the graph, $6=6$ ✓
   $C(4,-2)$ is on the graph, $6=6$ ✓
   Therefore, $|x|+|y|=6$ is probably the equation of the graph.

   Consider $y+3=9-|x|$

   $A(-3,3)$ is on the graph, $6=6$ ✓
   $B(1,5)$ is on the graph, $8=8$ ✓
   $C(4,-2)$ is not on the graph, $1\neq 5$ ✓
   Therefore, $y+3=9-|x|$ is not the equation of the graph.

