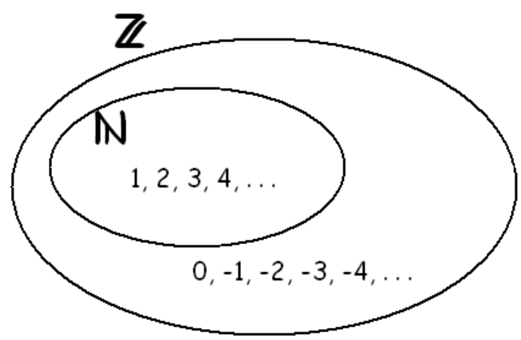
Recall that the set of all natural numbers (also called counting numbers) is $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. This set was historically the first set that people used. In this course, it will be a recurring theme that a mathematical system or set would be enlarged. This was the case with the natural numbers. Why would mathematicians of past centuries feel the need to step beyond the natural numbers? One reason is closure.

Recall the meaning of closure. The set $\mathbb{N}$ is closed under addition. In other words, the sum of *any* two natural numbers is also a natural number. $\mathbb{N}$ is also closed under multiplication. However, we do not have closure under subtraction and division. We *can* find a subtraction, say $3 - 12$, or a division, $10 \div 7$ that do not result in natural number. If we stay within the set of natural numbers, this means that the results for $3 - 12$ or $10 \div 7$ do not exist. It is a common theme in mathematics to work towards closure.

> The set of all natural numbers, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ is closed under addition and multiplication, but not under subtraction and division.

> **Definition:** The set of all integers, denoted by $\mathbb{Z}$, is the set
>
> $$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

The set of integers completely contains the set of natural numbers. In other words, the set of all natural numbers is a subset of the set of all integers, $\mathbb{N} \subseteq \mathbb{Z}$. We can also imagine that we started with the natural numbers and added zero and the opposite of each natural number to form the set of all integers.



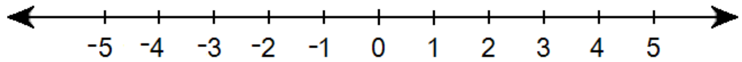
> **Definition:** The **opposite** of 3 is written as -3. For any number, the sum of the number and its opposite is zero. Another expression for the opposite is the **additive inverse**.

The opposite of 3 is -3. The opposite of -3 is 3. The opposite of zero is zero itself.

The negative sign already has a meaning, that of subtraction. We now are facing an ambiguity that is often the source of confusion. Does a negative sign denote the opposite of a number, or does it denote subtraction? This is a question that we often need to ask ourselves. While the answer always clearly exists, it very much depends on the context. For example, the negative sign in -3 clearly denotes that we are talking about the opposite of 3 or negative 3. However, if we place a number in front of it, like in $8 - 3$, the same sign here denotes subtraction. And what about $8(-3)$? Now the parentheses tells us that the negative sign does not denote subtraction, rather it describes the number after it as negative.

-3	$8 - 3$	$8(-3)$
the opposite of 3	subtraction	the opposite of 3

We often depict integers with a number line.



> **Definition:** (Ordering on $\mathbb{Z}$) Between two integers, the one on the right is greater.

Another way of envisioning this is to think of a positive number as money and a negative number as debt, and ask: who is richer? $2 < 10$ was obvious, and also that $-5 < 3$, but now we see that between -100 and -2, -2 is greater. After all, the person who has no money and only 2 dollars of debt is better off than another person who has no money and a 100 dollars of debt. And so $-100 < -2$.

We can swap inequalites, as long as the smaller part of the inequality sign points to the smaller number.

$-100 < -2$	read: -100 is less than -2	$-100 \leq -2$	read: -100 is less than or equal to -2
$-2 > -100$	read: -2 is greater than -100	$-2 \geq -100$	read: -2 is greater than or equal to -100

Wether we plot them on a number line or think money and debt, we will agree that $-1000\,000$ (negative one million) is less than 5. But what if we wanted to compare the size of numbers, ignoring their signs? Suppose we want to say that a million dollar debt is a lot of debt. In this case, we use the concept of the absolute value of a number.

> **Definition:** The **absolute value of a number** is its distance from zero on the number line. We denote the absolute value of a number x by $|\,x\,|$.

Distances can never be negative. -5 is 5 units away from zero on the number line. So is 5, only it is in the other direction. So, the absolute value of 5 and -5 are both 5.

Example 1. Compute each of the following.

a) $|-2|$ b) $|\,2\,|$ c) $|\,0\,|$ d) $-|-5|$

Solution: a) The number -2 is 2 units away from zero on the number line. Thus $|-2| = 2$.

b) The number 2 is 2 units away from zero on the number line. Thus $|\,2\,| = 2$.

c) The distance between zero and zero on the number line is zero. Thus $|\,0\,| = 0$.

d) This is an example, where two negatives do not make a positive. The way we can read this is: *the opposite of the absolute value of negative five*. The absolute value of -5 is 5. The opposite of that is -5. Using notation, $-|-5| = -5$.

Addition of Integers: Again, think money and debt. Positive numbers represent money, negative numbers represent debt. Adding zero to anything will leave the other number unchanged.

Example 2. Compute each of the following sums.

a) $-4+7$ b) $-3+(-8)$ c) $3+(-14)$ d) $-7+2$ e) $-2+0$

Solution: a) We think of $-4+7$ as follows. We start with a person who has no money and is in debt by 4 dollars. To this, we add 7 dollars. So the person pays off all that 4 dollar debt and is still left with 3 dollars. So $-4+7=3$.

b) We think of $-3+(-8)$ as follows. We start with a person who has no money and is in debt by 3 dollars. To this, we add another debt of 8 dollars. So this person is still now in debt by 11 dollars. So $-3+(-8)=-11$.

c) We think of $3+(-14)$ as follows. We start with a person who has 3 dollars. To this, we add a debt of 14 dollars. So the person pays off all the debt he can - that is 3 dollars and is still in debt by 11 dollars. So $3+(-14)=-11$.

d) We think of $-7+2$ as follows. We start with a person who has no money and is in debt by 7 dollars. Then this person gets 2 dollars. So the person pays off all the debt she can. After she pays off 2 dollars of debt, she has no money and is still left with 5 dollars of debt. So $-7+2=-5$.

e) Adding zero to any number leaves the other number unchanged. Therefore, $-2+0=-2$.

We already know how to add two positive numbers, and we know that the sum is positive. Adding two negative numbers is similar, we are summing debts. So we know that the sum of two negative numbers is also negative. If we add a negative and a positive number, the result may be positive or negative, depending on which number's size (or absolute value) is greater.

Adding zero to any number leaves that number unchanged. In other words, for any integer x, $x+0=x$ and $0+x=x$. In the language of algebra, we refer to a number that has no effect in an operation as an identity or identity element.

Definition: When added to any integer, zero has no effect. Because of this behavior, we call zero an **additive identity**.

Discussion: Based on its behavior, can you find a multiplicative identity within the set of all integers?

Now that we can add integers, we need to return to absolute values. The absolute value sign is also a grouping symbol that overwrites order of operations. So if there is a sum (or any other expression) within the absolute value sign, we need to perform those until we are left with just a number within the absolute value sign. Then we take the absolute value of that number.

Example 3. Compute each of the following.

a) $|-9+4|$ b) $|-9|+|4|$ c) $|8|+|-7|$ d) $|8+(-7)|$

Solution: a) The absolute value sign is also a pair of parentheses. We perform the addition $-9+4$ inside, and get -5. Then we take the absolute value of -5.
$|-9+4|=|-5|=5$

b) In this example we take the absolute values and then add.
$|-9|+|4|=9+4=13$

c) We take the absolute values and then add.
$|8|+|-7|=8+7=15$

d) We first perform the addition inside and then take the absolute value.
$|8+(-7)|=|1|=1$

Subtraction of Integers: the following statement is always true, and is often extremely useful.

To subtract is to add the opposite.

Of course, we don't always use this fact. In the subtraction $10 - 3$, we would only complicate things by applying this fact. It would still get us the right result. Instead of subtracting positive 3, we add its opposite, negative 3.

$$10 - 3 = 7 \quad \text{and also,} \quad 10 + (-3) = 7$$

Consider the subtraction $100 - (-20)$. We are asked to subtract negative 20. To subtract is to add the opposite. So, instead of subtracting negative 20, we will add its opposite, positive 20.

$$100 - (-20) = 100 + 20 = 120$$

Example 4. Compute each of the following.

a) $-7 - 8$ b) $-9 - (-5)$ c) $1 - 7$ d) $6 - (-3)$

Solution: a) First, the negative sign in front of the 7 cannot denote subtraction. We are asked to subtract positive 8 from negative 7. To subtract is to add the opposite. Instead of subtracting positive 8, we will add its opposite, negative 8.
$-7 - 8 = -7 + (-8) = -15$

b) To subtract is to add the opposite. Instead of subtracting negative 5, we will add its opposite, positive 5.
$-9 - (-5) = -9 + 5 = -4$

c) To subtract is to add the opposite. Instead of subtracting positive 7, we will add its opposite, negative 7.
$1 - 7 = 1 + (-7) = -6$

d) To subtract is to add the opposite. Instead of subtracting negative 3, we will add its opposite, positive 3.
$6 - (-3) = 6 + 3 = 9$

Why is $100 - (-20) = 100 + 20$? Even if we understand how to compute this, it would be nice to understand why this is correct. So here is one way to think about this.

Imagine that we have both a bank account a credit card with a bank. Suppose that at the moment, we have 150 dollars in the bank but we also owe 50 dollars to the bank on the credit card. So our net worth is 100 dollars.

Money in bank	Debt on credit card	Total Net worth
150	50	100

Suppose now that we have collected enough bonus points on the credit card to earn rewards. So the bank reduces our credit card debt by 20 dollars. (i.e. subtracts 20 debt, i.e. subtracts negative 20). We still have our 150 in cash, but now our debt is reduced to 30 dollars. So our net worth is now 120 dollars. That is 20 dollars more than before.

	Money in bank	Debt on credit card	Total Net worth
before	150	50	100
after	150	30	120

After all, reducing our debt by 20 dollars is almost the same as if someone gave us 20 dollars so that we can pay off some of our debts.

Multiplication of Integers: Multiply the absolute values. If two integers have the same sign, their product is positive. If two integers have different signs, their product is negative. If any of the factors is zero, the product is zero.

Example 5. Compute each of the following.

a) $-3 \cdot 5$ b) $-4(-5)$ c) $10(-2)$ d) $0(-3)$ e) $-1(8)$

Solution: a) The product of a negative and a positive number is negative.
$-3 \cdot 5 = -15$

b) The product of two negative numbers is positive.
$-4(-5) = 20$

c) The product of a positive and a negative number is negative.
$10(-2) = -20$

d) If any of the factors is zero, the product is zero.
$0(-3) = 0$

e) The product of a negative and a positive number is negative.
$-1(8) = -8$

Notice that if we multiply any integer by -1, the result is the opposite of that integer. This will be very useful later.

Why do these rules work this way? Here is one possible explanation. Multiplication is defined as repeated addition. For example, $4 \cdot 7$ means that we add 7 to itself, 4 times.

$$4 \cdot 7 = 7 + 7 + 7 + 7 = 28$$

Consider now $4 \cdot (-7)$. This means that we add -7 to itself, 4 times

$$4 \cdot (-7) = -7 + (-7) + (-7) + (-7) = -28$$

The logic becomes a bit tortured, but it also works with the first factor being negative. Consider now the product $-5 \cdot 8$. We can interpret the first negative sign as repeated subtraction. So, we are subtracting 8 repeatedly, 5 times. If we feel that we don't have anything to subtract the first 8 from, we can fix that by inserting a zero. We know that adding zero will not change any value.

$$\begin{aligned} -5 \cdot 8 &= -8 - 8 - 8 - 8 - 8 \\ &= 0 - 8 - 8 - 8 - 8 - 8 && \text{to subtract is to add the opposite} \\ &= 0 + (-8) + (-8) + (-8) + (-8) + (-8) \\ &= -40 \end{aligned}$$

The most interesting case is probably when we are multiplying two negative numbers. Consider the product $-4 \cdot (-10)$. The first negative sign is interpreted as repeated subtraction, the second one is that we are subtracting negative numbers. So we are subtracting negative 10 repeatedly, 4 times. If we need something to subtract the first negative 10 from, we will just insert a zero at the beginning.

$$\begin{aligned} -4 \cdot (-10) &= 0 - (-10) - (-10) - (-10) - (-10) && \text{to subtract is to add the opposite} \\ &= 0 + 10 + 10 + 10 + 10 = 40 \end{aligned}$$

Division of integers: We will deal with zero later. For the quotient of any two non-zero integers, the rules are very simple and similar to those of multiplication. Divide the absolute values. If the the integers have the same sign, the quotient is positive. If they have different signs, the quotient is negative.

Example 6. Compute each of the following.

a) $14 \div (-2)$ b) $-24 \div (-6)$ c) $-10 \div 5$

Solution: a) Divide the absolute values. The quotient of a positive and a negative number is negative.
$14 \div (-2) = -7$

b) Divide the absolute values. The quotient of two negative numbers is positive.
$-24 \div (-6) = 4$

c) Divide the absolute values. The quotient of a negative and a positive number is negative.
$-10 \div 5 = -2$

Division is often denoted by a horizontal bar. The same computations can also be written as $\dfrac{14}{-2} = -7$ and $\dfrac{-24}{-6} = 4$ and $\dfrac{-10}{5} = -2$.

Why do these rules work this way? Division is defined in terms of multiplication backward. In other words,

$$\frac{20}{4} = 5 \text{ is true because } 4 \cdot 5 = 20$$

Let us apply this principle. What is the result of $14 \div (-2)$?

$$\frac{14}{-2} = \boxed{?} \text{ would be true because } -2 \cdot \boxed{?} = 14$$

Since $-2 \cdot 7$ would result in -14, we can only choose -7 to make the multiplication backward work. $-2\,(-7) = 14$ and therefore $\dfrac{14}{-2} = -7$.

$$\text{Similarly, } \quad \frac{-24}{-6} = \boxed{?} \text{ would be true because } -6 \cdot \boxed{?} = -24$$

We need to multiply -6 by a positive number to get a negative product. So only positive 4 will work, thus $\dfrac{-24}{-6} = 4$.

Division by Zero: Now that we understand that division is defined in terms of multiplication backward, we can easily deal with zero. The expressions $\dfrac{0}{3}$ and $\dfrac{3}{0}$ look very similar, and yet they are very different.

$$\frac{0}{3} = \boxed{?} \text{ would be true because } 3 \cdot \boxed{?} = 0$$

In this case, we can only use zero to make the multiplication backward work. Let us investigate the other case.

$$\frac{3}{0} = \boxed{?} \text{ would be true because } 0 \cdot \boxed{?} = 3$$

Now we are in trouble. If we multiply any number by zero, the product is zero. Therefore, we can not meaningfully complete this division, and so we say that $\dfrac{3}{0}$ is undefined. What about $\dfrac{0}{0}$?

$$\frac{0}{0} = \boxed{?} \text{ would be true because } 0 \cdot \boxed{?} = 0$$

Now the problem is that *every* number would work, because any number times zero is zero. Mathematicians prefer one clean answer as a result of an operation. We do not like an operation that results in several numbers, let alone every number! So, one fundamental rule of mathematics is that division by zero is not allowed. Indeed, division by zero is not just an error: it is one of the worst errors.

The first commandment of mathemtics: *Thou shall not divide by zero. Ever...*

Changes in Notation

With the introduction of negative numbers, our notation will have to be modified. It is a widely accepted convention that if there are several signs (operations or negative) between two numbers, a pair of parentheses must separate them.

$-2+-6$	$-5--3$	$-3\cdot-4$	$-30\div-5$
$+-$ is not allowed	$--$ is not allowed	$\cdot -$ is not allowed	$\div-$ is not allowed

For this reason, until a few decades ago, we used to put a pair of parentheses around *every negative number*.

$(-2)+(-6)$	$(-5)-(-3)$	$(-3)\cdot(-4)$	$(-30)\div(-5)$
old style	old style	old style	old style

The development of mathematical notation is an ongoing process. The most important goal in notation is clarity. As long as clarity is not jeopardized, mathematicians are in the habit of omitting things. A few decades ago we stopped putting the parentheses around the first negative number in the line or inside a parentheses, because there was no risk that we would read the sign incorrectly as subtraction. Also, there is rarely an operation sign in front of the first number.

$-2+(-6)$	$-5-(-3)$	$-3\cdot(-4)$	$-30\div(-5)$
more modern	more modern	more modern	more modern

In the case of multiplication, we can omit one more thing. Recall that multiplication is the default operation; if we see two numbers with *nothing* between them, that indicates multiplication. For example, there is no operation sign or parentheses in $2x$ or ab and yet it is clear that the operation is multiplication. Now that most negative numbers must be placed in parentheses, we can often omit the dot indicating multiplication.

$-2+(-6)$	$-5-(-3)$	$-3\,(-4)$	$-30\div(-5)$
cannot omit anything	cannot omit anything	we can omit the dot	cannot omit anything

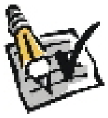
The most common modern style is minimalistic, ommitting as much as possible, as long as confusion is avoided. This can lead to apparent irregularities in notation. For example, our notation will be $2\,(-3)$, but when we swap the two factors, it will be $-3\cdot 2$.

Our Larger Number Sysytem

The set of all integers is closed under addition, subtraction, and multiplication. It is not closed under division.

As a matter of fact, the set of all integers ($\mathbb{Z}$) is the smallest set that contains the set of all natural numbers ($\mathbb{N}$) and is closed under subtraction. Another way to state this is that $\mathbb{Z}$ is the closure of $\mathbb{N}$ under subtraction.

In the future, we will further enlarge our number system to obtain closure under division.



Practice Problems

1. Label each of the following statements as true or false.

   a) $-3 \in \mathbb{Z}$ c) $\mathbb{Z} \subseteq \mathbb{N}$ e) $-|-2| = -2$

   b) $-3 \notin \mathbb{N}$ d) $-3 \geq -3$ f) For every integer x, $|x| \geq x$

2. Label each of the following statements as true or false.

   a) Every integer is a natural number.

   b) Every natural number is an integer.

   c) $3 < -5$ or $-2 > -8$

   d) $3 < -5$ and $-2 > -8$

   e) Zero is also called the additive identity.

   f) $5 \leq 5$ and $-|-8| = -8$

   g) $-2 < -2$ or $|-2| > |-10|$

3. Place an inequality sign between the given numbers to make the statement true.

   a) $5 \quad -7$ b) $-12 \quad -4$ c) $0 \quad -8$ d) $-1 \quad -4$ e) $-7 \quad -7$

4. Simplify each of the following.

   a) $|-5|$ b) $|5|$ c) $-|5|$ d) $-|-5|$ e) $|0|$ f) $|-12+9|$ g) $|-12| + |9|$

5. Perform the indicated operations.

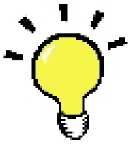
   a) $-2+7$ b) $-7-(-4)$ c) $12 \div (-2)$ d) $5(-3)$

   e) $-8 \cdot 0$ f) $-3-(-10)$ g) $-20 \div 0$ h) $-12 \div 3$

   i) $-4 \cdot 7$ j) $-6-|-7|$ k) $-3 \div (-3)$ l) $|9| + (-1)$

   m) $-3-0$ n) $0(-4)$ o) $\dfrac{-5}{0}$

   p) $0 \div (-1)$ q) $9+|-1|$ r) $|9+(-1)|$



Answers for Practice Problems

1. a) true b) true c) false d) true e) true f) true

2. a) false b) true c) true d) false e) true f) true g) false

3. a) $5 > -7$ or $5 \geq -7$ b) $-12 < -4$ or $-12 \leq -4$ c) $0 > -8$ or $0 \geq -8$ d) $-1 > -4$ or $-1 \geq -4$ e) $-7 \geq -7$ or $-7 \leq -7$

4. a) 5 b) 5 c) -5 d) -5 e) 0 f) 3 g) 21

5. a) 5 b) -3 c) -6 d) -15 e) 0 f) 7 g) undefined h) -4 i) -28 j) -13 k) 1 l) 8 m) -3 n) 0 o) undefined p) 0 q) 10 r) 8

