Equations are a fundamental concept and tool in mathematics.

> **Definition**: An **equation** is a statement in which two expressions (algebraic or numeric) are connected with an equal sign.

For example, $3x^2 - x = 4x + 28$ is an equation. So is $x^2 + 5y = -y^2 + x + 2$.

> **Definition**: A **solution** of an equation is a number (or an ordered set of numbers) that, when substituted into the variable(s) in the equation, makes the statement of equality true.

Example 1. a) Verify that -2 is not a solution of the equation $3x^2 - x = 4x + 28$.

b) Verify that 4 is a solution of the equation $3x^2 - x = 4x + 28$.

Solution: a) Consider the equation $3x^2 - x = 4x + 28$ with $x = -2$. We substitute $x = -2$ into both sides of the equation and evaluate the expressions.

If $x = -2$, the left-hand side of the equation is $$\begin{aligned} \text{LHS} &= 3x^2 - x \\ &= 3(-2)^2 - (-2) \\ &= 3 \cdot 4 + 2 = 12 + 2 = 14 \end{aligned}$$	If $x = -2$, the right-hand side of the equation is $$\begin{aligned} \text{RHS} &= 4x + 28 \\ &= 4(-2) + 28 \\ &= -8 + 28 = 20 \end{aligned}$$

Since the two sides are not equal, $14 \neq 20$, the number -2 is not a solution of this equation.

b) Consider the equation $3x^2 - x = 4x + 28$ with $x = 4$. We evaluate both sides of the equation after substituting 4 into x.

If $x = 4$, the left-hand side of the equation is $$\begin{aligned} \text{LHS} &= 3x^2 - x \\ &= 3 \cdot 4^2 - 4 \\ &= 3 \cdot 16 - 4 = 48 - 4 = 44 \end{aligned}$$	If $x = 4$, the right-hand side of the equation is $$\begin{aligned} \text{RHS} &= 4x + 28 \\ &= 4 \cdot 4 + 28 \\ &= 16 + 28 = 44 \end{aligned}$$

Since the two sides are equal, $x = 4$ is a solution of this equation.

> **Definition**: To **solve an equation** is to find *all* solutions of it. The set of all solutions is also called the solution set.

Caution! Finding one solution for an equation is not the same as solving it. For example, the number 2 is a solution of the equation $x^3 = 4x$. However, -2 is also a solution of this equation.

If we think about it a little, trial and error is never a legitimate method because there is no way for us to gurantee that there are no other solutions are there. It is impossible for us to try all real numbers because there are infinitely many of them, and we have finite lives.

So we will need to develop systematic methods to solve equations.

We will start with the easiest group of equations, linear equations. There are several types of linear equations, and we will start with the easiest type that is called one-step equations.

To solve a linear equation, we isolate the unknown by applying the same operation(s) to both sides. Consider, for example, Ann and Dewitt who has the same monthly salary. This month they both get a 40 dollar raise. Who is making ore money now? It is clear that if we start with two equal quantities and we add the same amount to them, they will still stay equal. This is the underlying principle of solving equations. We always apply the same operations to both sides in an effort to bring the equation in a simple form such as $x=-2$. The following equations are **one-step equations** because there is only one operation that separates us from the desired form.

Example 2. Solve each of the given equations. Make sure to check your solutions.

a) $x-8=10$ b) $3y=-12$ c) $\dfrac{x}{-3}=8$ d) $m+10=-5$

Equations like these are called **one-step equations** because they can be solved in only one step. We need to isolate the unknown on one side. In order to do that, we perform the inverse operation. (The inverse operation of additon is subtraction and vica versa. The inverse operation of multiplication is division and vica versa.)

Solution: a) In order to isolate the unknown, we add 8 to both sides.

$$\begin{aligned} x-8 &= 10 \qquad \text{add } 8 \\ x &= 18 \end{aligned}$$

So the only solution of this equation is 18. We can also say that the solution set is $\{18\}$. We should check; if $x=18$, the left-hand side is

$$\text{LHS} = x-8 = 18-8 = 10 = \text{RHS} \checkmark$$

So our solution, $\boxed{x=18}$ is correct.

b) In order to isolate the unknown, we divide both sides by 3.

$$\begin{aligned} 3y &= -12 \qquad \text{dividde by } 3 \\ y &= -4 \end{aligned}$$

So the only solution of this equation is -4. We check; if $y=-4$, then

$$\text{LHS} = 3y = 3(-4) = -12 = \text{RHS} \checkmark$$

So our solution, $\boxed{y=-4}$ is correct.

c) In order to isolate the unknown, we multiply both sides by -3.

$$\begin{aligned} \frac{x}{-3} &= 8 \qquad \text{multiply by } -3 \\ x &= -24 \end{aligned}$$

So the only solution of this equation is -24. We check; if $x=-24$, then

$$\text{LHS} = \frac{-24}{-3} = 8 = \text{RHS} \checkmark$$

So our solution, $\boxed{x=-24}$ is correct.

d) In order to isolate the unknown, we we subtract 10 from both sides.

$$\begin{aligned} m+10 &= -5 \qquad \text{subtract } 10 \\ m &= -15 \end{aligned}$$

So the only solution of this equation is -15. We check; if $m=-15$, then

$$\text{LHS} = m+10 = -15+10 = -5 = \text{RHS}$$

So our solution, $\boxed{m=-15}$ is correct.

Discussion: Solve each of the following equations. How are these unusual?

a) $5x = 5$ b) $5x = 0$ c) $x-4 = -4$ d) $\dfrac{x}{3} = 0$

Example 3. One side of a rectangle is 12 feet long. Find the length of the other side if the area of the rectangle is 60 square-feet.

Solution: Let us denote the missing side by x. We will write and solve an equation expressing the area of the rectangle.

$$\begin{aligned} 12x &= 60 \qquad \text{divide by } 12 \\ x &= 5 \end{aligned}$$

Thus the other side is $\boxed{5 \text{ feet}}$ long. Note that if we carry the units in the computation, they will work out perfectly.

$$(12\,\text{ft})\,x = 60\,\text{ft}^2 \qquad \text{divide by } 12\,\text{ft} \qquad \text{margin work:} \quad \frac{60\,\text{ft}^2}{12\,\text{ft}} = 5\frac{\text{ft}\cdot\text{ft}}{\text{ft}} = 5\,\text{ft}$$

$$x = 5\,\text{ft}$$

Part 2 – Two-Step Equations

Suppose we decide to hide a small object, say a coin. We put the coin on the table, then place an envelope over it, and then, just to be sure, we place a hat on top of the envelope. Let us find the coin! To do that, what do we need to remove, and in what order? We would first remove the hat and then the envelope, right?

This is the basis of solving two-step equations. To isolate the unknown, we will perform the inverse operations, in the reverse order. What happened last can be undone first.

Example 4. Solve each of the given equations. Make sure to check your solutions.

a) $10 = 3x-11$ b) $3x+8 = -7$ c) $\dfrac{t-7}{2} = -8$ d) $\dfrac{x}{-3}+4 = 15$

Solution: a) The equation $10 = 3x - 11$ looks unusual in the sense that two-step equations often contain the unknown on the left-hand side. We are always allowed to swap two sides of an equation. If $A = B$, then clearly, also $B = A$. We will do this first. This is an optional step that is always available.

$$\begin{aligned} 10 &= 3x - 11 && \text{we swap the two sides} \\ 3x - 11 &= 10 \end{aligned}$$

We now look at the side that contains x and ask: *What happened to the unknown*? The answer is: *Multiplication by* 3 *and then subtraction of* 11. We need to apply the inverse operations, in reverse order. In this case, this means that we will add 11 to both sides and then divide both sides by 3.

$$\begin{aligned} 3x - 11 &= 10 && \text{add } 11 \\ 3x &= 21 && \text{divide by } 3 \\ x &= 7 \end{aligned}$$

So the only solution of this equation is 7. We check: if $x = 7$, then

$$\text{LHS} = 3x - 11 = 3 \cdot 7 - 11 = 21 - 11 = 10 = \text{RHS} \checkmark$$

So our solution, $\boxed{x = 7}$ is correct.

b) As we look at the equation $3x + 8 = -7$, we ask: *What happened to the unknown*? The answer is: *Multiplication by* 3 *and then addition of* 8. We need to apply the inverse operations, in a reverse order. In this case, this means that we will subtract 8 from both sides and then divide both sides by 3.

$$\begin{aligned} 3x + 8 &= -7 && \text{subtract } 8 \\ 3x &= -15 && \text{divide by } 3 \\ x &= -5 \end{aligned}$$

So the only solution of this equation is -5. We check: if $x = -5$, then

$$\text{LHS} = 3x + 8 = 3(-5) + 8 = -15 + 8 = -7 = \text{RHS} \checkmark$$

So our solution, $\boxed{x = -5}$ is correct.

c) What happened to the unknown? On the left-hand side, there was a subtraction of 7 and then a division by 2. To reverse that, we will multiply both sides by 2 and then add 7 to both sides.

$$\begin{aligned} \frac{t - 7}{2} &= -8 && \text{multiply by } 2 \\ t - 7 &= -16 && \text{add } 7 \\ t &= -9 \end{aligned}$$

So the only solution of this equation is -9. We check: if $t = -9$, then

$$\text{LHS} = \frac{t - 7}{2} = \frac{-9 - 7}{2} = \frac{-16}{2} = -8 = \text{RHS} \checkmark$$

So our solution, $\boxed{t = -9}$ is correct.

d) What happened to the unknown? On the left-hand side, there was a division by -3 and then an addition of 4. To reverse that, we will subtract 4 from both sides by and then multiply both sides by -3.

$$\begin{aligned} \frac{x}{-3}+4 &= 15 && \text{subtract } 4 \\ \frac{x}{-3} &= 11 && \text{multiply by } -3 \\ x &= -33 \end{aligned}$$

So the only solution of this equation is -33. We check: if $x=-33$, then

$$\text{LHS} = \frac{x}{-3}+4 = \frac{-33}{-3}+4 = 11+4 = 15 = \text{RHS} \checkmark$$

So our solution, $\boxed{x=-33}$ is correct.

Example 5. The sum of three times a number and seven is -5. Find this number.

Solution: Let us denote our mystery number by x. The equation will be just the first sentence, translated to algebra. The sum of three times the number and seven is $3x+7$. So our equation is $3x+7=-5$. We will know the number if we solve this equation.

$$\begin{aligned} 3x+7 &= -5 && \text{subtract } 7 \\ 3x &= -12 && \text{divide by } 3 \\ x &= -4 \end{aligned}$$

Good news! We do not need to check if -4 is indeed the solution of the equation. What if we correctly solved the *wrong* equation? Recall that *we* came up with the equation, it was not given. Instead of checking the number against the equation, we should check if our solution satisfies the conditions stated in the problem. Is it true the sum of three times -4 and seven is -5? Indeed,
$3(-4)+7=-12+7=-5$. Thus our solution, $\boxed{-4}$, is correct.

Sample Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x-5=17$
2. $\dfrac{a-10}{5}=-3$
3. $\dfrac{t}{4}-10=-4$
4. $\dfrac{t-5}{12}=4$
5. $2x-7=-3$
6. $\dfrac{x+8}{3}=-2$
7. $\dfrac{x}{3}+8=-2$
8. $-2x+3=3$
9. $3(x+7)=36$
10. $3x-10=-10$
11. $-4x+6=-18$

Solve each of the following application problems.

12. Paul invested his money on the stock market. First he bet on a risky stock and lost half of his money. Then he became a bit more careful and invested money in more conservative stocks that involved less risk but also less profit. His investments made him 80 dollars. If he has 250 dollars in the stock market today, with how much money did he start investing?
13. In a hotel, the first night costs 45 dollars, and all additional nights cost 35 dollars. How long did Mr. Williams stay in the hotel if his bill was 325 dollars?

Practice Problems

Solve each of the following equations. Make sure to check your solutions.

14. $2x-3=-11$
15. $-2x-3=7$
16. $5x-3=17$
17. $\dfrac{x-3}{7}=-2$
18. $\dfrac{x}{7}-3=-1$
19. $-4x-3=13$
20. $\dfrac{a+1}{4}=-9$
21. $5x-6=-6$
22. $\dfrac{x}{7}-1=-3$
23. $-x+5=-7$
24. $\dfrac{2x-1}{7}=-3$
25. $5(x-2)=-20$
26. $\dfrac{x-8}{7}=-2$
27. $3b+13=-5$
28. $\dfrac{x}{3}-7=7$

Solve each of the following application problems.

16. Three times the difference of x and 7 is -15. Find x.
17. Ann and Bonnie are discussing their financial situation. Ann said: *If you take* 50 *bucks from me and then doubled what is left, I would have* \$300. Bonnie answers: *That's funny. If you doubled my money first and then took* \$50, *then I would have* \$300! How much do they each have?
18. Susan was asked about her age. She answered as follows: My big brother's age is six less than three times my age. How old is Susan if her big brother is 21 years old?

Answers

Discussion

a) 1 b) 0 c) 0 d) 0

One thing that is unusual in this problem is the idea of cancellation. Cancellation results in 0 or 1, depending on the operation.

Sample Problems

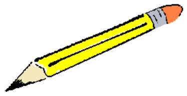
19. 11 2. -5 3. 24 4. 53 5. 2 6. -14 7. -30 8. 0 9. 5 10. 0 11. -4

12. \$340 13. 9 nights

Practice Problems

20. -4 2. -5 3. 4 4. -11 5. 14 6. -4 7. -37 8. 0 9. -14 10. 12

11. -10 12. -2 13. -6 14. -6 15. 42 16. 2 17. Ann has \$200 and Bonnie has \$175



Sample Problems - Solutions

Solve each of the following equations. Make sure to check your solutions.

1. $2x - 5 = 17$

   Solution:

$$\begin{aligned} 2x - 5 &= 17 && \text{add 5 to both sides} \\ 2x &= 22 && \text{divide by 2} \\ x &= 11 \end{aligned}$$

   We check: if $x = 11$, then

$$\text{RHS} = 2(11) - 5 = 22 - 5 = 17 = \text{LHS}$$

   Thus our solution, $\boxed{x = 11}$ is correct.

2. $\dfrac{a - 10}{5} = -3$

   Solution:

$$\begin{aligned} \frac{a-10}{5} &= -3 && \text{multiply both sides by 5} \\ a - 10 &= -15 && \text{add 10 to both sides} \\ a &= -5 \end{aligned}$$

   We check: if $a = -5$, then

$$\text{LHS} = \frac{-5 - 10}{5} = \frac{-15}{5} = -3 = \text{RHS}$$

   Thus our solution, $\boxed{a = -5}$ is correct.

3. $\dfrac{t}{4} - 10 = -4$

   Solution:

$$\begin{aligned} \frac{t}{4} - 10 &= -4 && \text{add 10 to both sides} \\ \frac{t}{4} &= 6 && \text{multiply both sides by 4} \\ t &= 24 \end{aligned}$$

   We check: if $t = 24$, then

$$\text{RHS} = \frac{t}{4} - 10 = \frac{24}{4} - 10 = 6 - 10 = -4 = \text{LHS}$$

   Thus our solution, $\boxed{t = 24}$ is correct.

4. $\dfrac{t-5}{12} = 4$

Solution:

$$\begin{aligned} \frac{t-5}{12} &= 4 && \text{multiply both sides by 12} \\ t-5 &= 48 && \text{add 5 to both sides} \\ t &= 53 \end{aligned}$$

We check: if $t = 53$, then $\text{LHS} = \dfrac{53-5}{12} = \dfrac{48}{12} = 4 = \text{RHS}.$

Thus our solution, $\boxed{t = 53}$ is correct.

5. $2x - 7 = -3$

Solution: We apply all operations to both sides.

$$\begin{aligned} 2x-7 &= -3 && \text{add } 7 \\ 2x &= 4 && \text{divide by 2} \\ x &= 2 \end{aligned}$$

We check: if $x = 2$, then

$$\text{LHS} = 2(2) - 7 = 4 - 7 = -3 = \text{RHS}$$

Thus our solution, $\boxed{x = 2}$ is correct.

6. $\dfrac{x+8}{3} = -2$

Solution: We apply all operations to both sides.

$$\begin{aligned} \frac{x+8}{3} &= -2 && \text{multiply by } 3 \\ x+8 &= -6 && \text{subtract } 8 \\ x &= -14 \end{aligned}$$

We check: $\text{LHS} = \dfrac{-14+8}{3} = \dfrac{-6}{3} = -2 = \text{RHS}$

Thus our solution, $\boxed{x = -14}$ is correct.

7. $\dfrac{x}{3} + 8 = -2$

Solution: We apply all operations to both sides.

$$\begin{aligned} \frac{x}{3} + 8 &= -2 && \text{subtract } 8 \\ \frac{x}{3} &= -10 && \text{multiply by } 3 \\ x &= -30 \end{aligned}$$

We check:

$$\text{LHS} = \frac{-30}{3} + 8 = -10 + 8 = -2 = \text{RHS}$$

Thus our solution, $\boxed{x = -30}$ is correct.

8. $-2x+3=3$

Solution: We apply all operations to both sides.

$$\begin{array}{rcll} -2x+3 &=& 3 & \text{subtract } 3 \\ -2x &=& 0 & \text{divide by } -2 \\ x &=& 0 & \end{array}$$

We check: if $x=0$, then

$$\text{LHS} = -2\cdot 0+3 = 0+3 = 3 = \text{RHS}$$

Thus our solution, $\boxed{x=0}$ is correct.

9. $3(x+7)=36$

Solution: We apply all operation to both sides,

$$\begin{array}{rcll} 3(x+7) &=& 36 & \text{divide by } 3 \\ x+7 &=& 12 & \text{subtract } 7 \\ x &=& 5 & \end{array}$$

We check: if $x=5$, then

$$\text{LHS} = 3(5+7) = 3\cdot 12 = 36 = \text{RHS}$$

Thus our solution, $\boxed{x=5}$ is correct.

10. $3x-10=-10$

Solution:

$$\begin{array}{rcll} 3x-10 &=& -10 & \text{add } 10 \text{ to both sides} \\ 3x &=& 0 & \text{divide by } 3 \\ x &=& 0 & \end{array}$$

We check: if $x=0$, then

$$\text{LHS} = 3\cdot 0-10 = 0-10 = -10 = \text{RHS}$$

Thus our solution, $\boxed{x=0}$ is correct.

11. $-4x+6=-18$

Solution:

$$\begin{array}{rcll} -4x+6 &=& -18 & \text{subtract } 6 \\ -4x &=& -24 & \text{divide by } -4 \\ x &=& 6 & \end{array}$$

We check:

$$\text{RHS} = -4x+6 = -4\cdot 6+6 = -24+6 = -18 = \text{LHS}$$

Thus our solution, $\boxed{x=6}$ is correct.

12. Paul invested his money on the stock market. First he bet on a risky stock and lost half of his money. Then he became a bit more careful and invested money in more conservative stocks that involved less risk but also less profit. His investments made him 80 dollars. If he has 250 dollars in the stock market today, with how much money did he start investing?

    Solution: Let us denote the amount of money with which Paul started to invest by x. First he lost half of his money, so he had $\frac{x}{2}$. Then he gained 80 dollars end ended up with 250 dollars. So, we can write the equation $\frac{x}{2}+80=250$. We will solve this two-step equation for x. What happened to the unknown was first division by 2 and then addition of 80. To reverse these operations, we will first subtract 80 and then multiply by 2.

$$\begin{aligned} \frac{x}{2}+80 &= 250 && \text{subtract 80} \\ \frac{x}{2} &= 170 && \text{multiply by 2} \\ x &= 340 \end{aligned}$$

    So Paul started with 340 dollars. We check: If we lose half of 340,dollars we have 170 dollars left. Then when we add 80,dollars we indeed end up with 250 dollars. So our solution is correct, $\boxed{\text{Paul started with 340 dollars}}$.

13. In a hotel, the first night costs 45 dollars, and all additional nights cost 35 dollars. How long did Mr. Williams stay in the hotel if his bill was 325 dollars?

    Solution: Suppose that Mr. Williams stayed for the first night and the an additional x many nights. Then the bill would be $45+x\cdot 35$ or $35x+45$. So we write and then solve the equation $35x+45=325$.

$$\begin{aligned} 35x+45 &= 325 && \text{subtract 45} \\ 35x &= 280 && \text{divide by 35} \\ x &= 8 \end{aligned}$$

    Thus Mr. Williams stayed in the hotel for 9 nights. Why not 8 if we got $x=8$? Remember, the first night was counted separately; there was the first night and then $x=8$ additional nights. This is why it is a good idea to read the text of the problem one more time before we state our final answer. So $\boxed{\text{Mr. Williams stayed 9 nights in the hotel}}$. We check: the bill for 9 nights would be $45+8(35)=325$, and so our solution is correct.