The Pythagorean theorem is a fundamental theorem about the connection between the three sides of a right triangle. Let us recall first a few things we will need for this topic.

Standard labeling simplifies notation. According to standard labeling, side a is opposite point A and angle α. Similarly, side b is opposite of point B and angle β, and side c is opposite point C and angle γ.

The three angles in any triangle add up to $180°$.

In any triangle, the order between sides is the same as the order between the angles opposite them. For example, the longest side is always opposite the greatest angle, and vica versa: the greatest angle is opposite the longest side. The shortest side is opposite the smallest angle, and vica versa: the smallest angle is opposite of the shortest side. If two sides are equally long, the angles opposite them are also equal.

In a right triangle, the right angle is the single greatest angle in the triangle, because the other two angles must add up to $90°$ so they both are smaller than $90°$.

Therefore, right triangles have a single longest side and it is located opposite of the right angle.

Definition: In a right triangle, the side opposite the right angle is the longest side. We call this side the **hypotenuse** of the triangle.

We are now ready to state the Pythagorean theorem. It actually has two parts.

Theorem: (The Pythagorean theorem)

Part 1. If ABC is a right triangle with shorter sides a and b and hypotenuse c, then

$$a^2 + b^2 = c^2$$

Part 2. In any triangle with sides a, b, and c, if $a^2 + b^2 = c^2$, then the angle opposite side c measures $90°$.

The first part tells us how right triangles behave. The second part tells us that *only* right triangles have this behavior.

Example 1. In each case, determine whether the three sides given are sides in a right triangle or not.

a) $5\,\mathrm{cm}$, $7\,\mathrm{cm}$, and $9\,\mathrm{cm}$ b) $5\,\mathrm{ft}$, $13\,\mathrm{ft}$ and $12\,\mathrm{ft}$ c) 53 units, 28 units, and 45 units.

Solution: We will use the second part of the Pythagorean theorem. Let us denote the longest side by c and the other two sides by a and b. If the statement $a^2 + b^2 = c^2$, is true, the triangle is a right triangle. If not, the triangle has no right angle in it.

a) The only side here that could be the hypotenuse, is the longest one, measuring $9\,\mathrm{cm}$. Therefore, the two quantities we need to compare are $(5\,\mathrm{cm})^2 + (7\,\mathrm{cm})^2$ and $(9\,\mathrm{cm})^2$.

$(5\,\mathrm{cm})^2 + (7\,\mathrm{cm})^2 = 25\,\mathrm{cm}^2 + 49\,\mathrm{cm}^2 = 74\,\mathrm{cm}^2$ and $(9\,\mathrm{cm})^2 = 81\,\mathrm{cm}^2$

Since $74\,\mathrm{cm}^2 \neq 81\,\mathrm{cm}^2$, this triangle is **not a right triangle.**

b) The only side here that could be the hypotenuse, is the longest one, measuring $13\,\mathrm{ft}$. Therefore, the two quantities we need to compare are $(5\,\mathrm{ft})^2 + (12\,\mathrm{ft})^2$ and $(13\,\mathrm{ft})^2$.

$(5\,\mathrm{ft})^2 + (12\,\mathrm{ft})^2 = 25\,\mathrm{ft}^2 + 144\,\mathrm{ft}^2 = 169\,\mathrm{ft}^2$ and $(13\,\mathrm{ft})^2 = 169\,\mathrm{ft}^2$

Since $169\,\mathrm{ft}^2 = 169\,\mathrm{ft}^2$, this triangle is **a right triangle** with hypotenuse $13\,\mathrm{ft}$ long.

c) The only side here that could be the hypotenuse, is the longest one, measuring 53 units. Therefore, the two quantities we need to compare are $28^2 + 45^2$ and 53^2.

$28^2 + 45^2 = 784 + 2025 = 2809$ and $53^2 = 2809$

Since $2809 = 2809$, this triangle is **a right triangle.**

If we know two sides of a right triangle, the Pythagorean theorem enables us to find the third side.

Example 2. In each case, find the missing side of the right triangle.

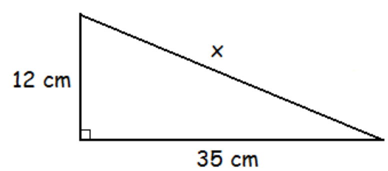
a) the shortest two sides are $12\,\mathrm{cm}$ and $35\,\mathrm{cm}$

b) the longest two sides are $15\,\mathrm{ft}$ and $17\,\mathrm{ft}$

Solution: a) We will use the first part of the Pythagorean theorem. Let us denote the hypotenuse by x.

We state the Pythagorean theorem for this right triangle.

$$12^2 + 35^2 = x^2$$
$$144 + 1225 = x^2$$
$$1369 = x^2$$
$$\pm 37 = x$$

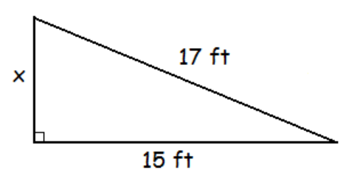


Since x represents a distance, and distances can never be negative, $x = -37$ is ruled out as a possible solution. Thus the hypotenuse of this triangle is $\boxed{37\,\mathrm{cm}}$.

b) We will use the first part of the Pythagorean theorem. Let us denote the missing shortest side by x.

We state the Pythagorean theorem for this right triangle.

$$x^2 + 15^2 = 17^2$$
$$x^2 + 225 = 289$$
$$x^2 = 64$$
$$x = \pm 8$$

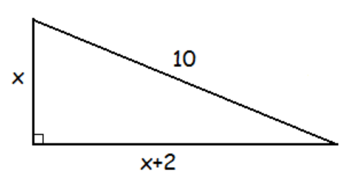


Since distances cannot be negative, $x = -8$ is ruled out as a possible solution. Thus the missing side is $\boxed{8\,\mathrm{ft}}$ long.

In order to find missing sides, we do not always need to know the lengths of two sides if other information is given.

Example 3. The hypotenuse of a right triangle is 10 units long. The difference between the other two sides is 2 units. Find the missing sides of the right triangle.

Solution: If we label the shortest side by x, then the other missing side can be denoted by $x + 2$. We state the Pythagorean theorem for this right triangle and solve the equation for x.



$x^2 + (x+2)^2 = 10^2$ expand complete square

$x^2 + x^2 + 4x + 4 = 10$ combine like terms

$2x^2 + 4x + 4 = 100$ subtract 100

$2x^2 + 4x - 96 = 0$ factor out 2

$2\left(x^2 + 2x - 48\right) = 0$ factor (we used trial and error)

$2\,(x+8)\,(x-6) = 0$ apply the zero product rule

$x_1 = -8, \quad x_2 = 6$

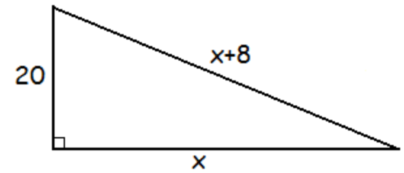
Since distances can never be negative, -8 is ruled out as a possible solution. The other solution, $x = 6$ means that the other side, denoted by $x+2$ must be $6+2 = 8$ units long. Thus, this right triangle has sides $\boxed{\text{6, 8, and 10 units}}$ long.

We check: $6^2 + 8^2 = 36 + 64 = 100 = 10^2$, thus the triangle is indeed right. Also, $8 - 6 = 2$, so our answer is correct.

The following example will be a similar application problem, but the computation will be entirely different.

Example 4. The shortest side of a right triangle is 20 units long. The difference between the other two sides is 8 units. Find the missing sides of the right triangle.

Solution: If we label the shorter missing side by x, then the other missing side, the hypotenuse can be denoted by $x + 8$. We state the Pythagorean theorem for this right triangle and solve the equation for x.



$$\begin{aligned} 20^2 + x^2 &= (x+8)^2 && \text{expand complete square} \\ 400 + x^2 &= x^2 + 16x + 64 && \text{subtract } x^2 \\ 400 &= 16x + 64 && \text{subtract 64} \\ 336 &= 16x && \text{divide by 16} \\ 21 &= x && \end{aligned}$$

Therefore, the shoter missing side, denoted by x is 21 units long, and the hypotenuse, denoted by $x + 8$ must be $21 + 8 = 29$ units long. So the three sides of this right triangle are $\boxed{\text{20, 21, and 29 units}}$ long. We check: the difference between 29 and 21 is indeed 8. For the Pythagorean theorem, $20^2 + 21^2 = 400 + 441 = 841$ and $29^2 = 841$, and so $20^2 + 21^2 = 29^2$. This means that this is indeed a right triangle, and so our answer is correct.

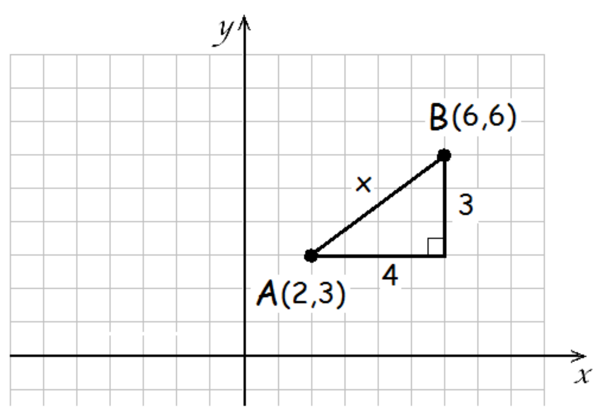
This problem turned out to be much easier because the equation was linear after x^2 was subtracted from both sides.

The following example is an extremely important application of the Pythagorean theorem.

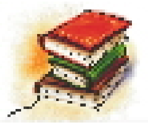
Example 5. Find the distance between the points $A(2,3)$ and $B(6,6)$.

Solution: Let us plot the given points in a coordinate system as shown on the picture. The right triangle on the picture has shorter sides 4 and 3 units long. We denote the line segment connecting A and B (the hypotenuse) by x and state the Pythagorean theorem.

$$\begin{aligned} 4^2 + 3^2 &= x^2 \\ 16 + 9 &= x^2 \\ 25 &= x^2 \\ \pm 5 &= x \end{aligned}$$



Since distances can never be negative, -5 is ruled out and so our answer is $\boxed{\text{5 units}}$.



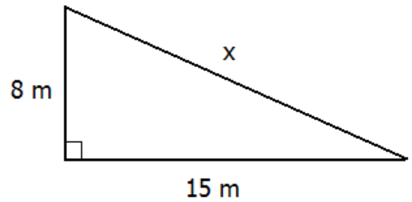
Sample Problems

1. Could the three line segments given below be the three sides of a right triangle? Explain your answer.

   a) $6\,\mathrm{cm}$, $10\,\mathrm{cm}$, and $8\,\mathrm{cm}$

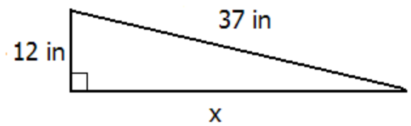
   b) $7\,\mathrm{ft}$, $15\,\mathrm{ft}$, and $50\,\mathrm{ft}$

   c) $4\,\mathrm{m}$, $5\,\mathrm{m}$, and $6\,\mathrm{m}$



2. Find the hypotenuse of the triangle shown on the figure.

3. Find the missing leg of the right triangle shown on the picture below.

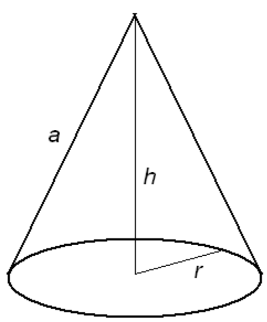


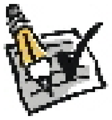
4. Find the distance between the points $(3, 8)$ and $(8, -4)$.

5. The sides of an isosceles triangle are 42 units, 29 units, and 29 units long. Find the length of the height drawn to the 42 units long side.

6. The hypotenuse of a right triangle is $20\,\mathrm{cm}$. The difference between the other two sides is $4\,\mathrm{cm}$. Find the sides of the triangle.

7. Find the height h of the cone shown on the picture below, if the base has a radius of $10\,\mathrm{m}$ and $a = 26\,\mathrm{m}$.



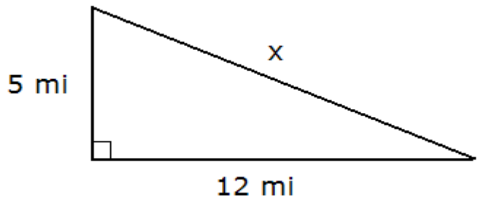


Practice Problems

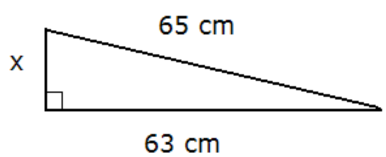
1. Could the three line segments given below be the three sides of a right triangle? Explain your answer.

   a) 3 cm, 7 cm, and 8 cm b) 37 ft, 12 ft, and 35 ft c) 6 m, 7 m, and 8 m

2. Find the hypotenuse of the triangle shown on the figure below.



3. Find the missing leg of the right triangle shown on the picture below.



4. The sides of an isosceles triangle are 25 m, 25 m, and 14 m long. Find the length of the height drawn to the 14 m long side.

5. Find the distance between the given points.

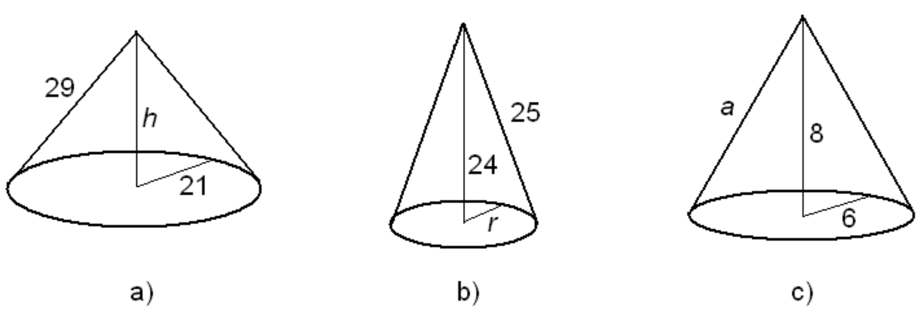
   a) $(-2,-3)$ and $(6,3)$ b) $(-9,-3)$ and $(15,4)$.

6. One leg of a right triangle is 9 cm. The difference between the other two sides is 1 cm. Find the length of all sides.

7. The hypotenuse of a right triangle is 50 in. The difference between the other two sides is 34 in. Find the length of all sides.

8. The shortest side of a right triangle is 16 units long. The difference between the other two sides is 4 units. Find the sides of this right riangle.

9. Find the missing lengths indicated on the picture below.



Answers

Sample Problems

1. a) There is a right angle opposite the $10\,\mathrm{cm}$ long side. b) not a right triangle

2. $17\,\mathrm{m}$ 3. 35 inches 4. 13 units 5. 20 units 6. $32\,\mathrm{cm}$ and $60\,\mathrm{cm}$ 7. $24\,\mathrm{m}$

Practice Problems

1. a) not a right triangle b) There is a right angle opposite the $37\,\mathrm{ft}$ long side. c) not a right triangle

2. $13\,\mathrm{mi}$ 3. $16\,\mathrm{cm}$ 4. $24\,\mathrm{m}$ 5. a) 10 units b) 25 units 6. $9\,\mathrm{cm}$, $40\,\mathrm{cm}$, and $41\,\mathrm{cm}$

7. $14\,\mathrm{in}$, $48\,\mathrm{in}$, and $50\,\mathrm{in}$ 8. 16, 30, and 34 units 9. a) 20 b) 7 c) 10

Sample Problems Solutions

1. Could the three line segments given be the three sides of a right triangle? Explain your answer.

   a) 6 cm, 10 cm, and 8 cm

   Solution: The longest side is 10 cm long. Thus, only this side can be the hypotenuse. We use the Pythagorean theorem to check for a right angle:

   $$6^2 + 8^2 \stackrel{?}{=} 10^2$$

   We get that the two quantities are equal, thus this triangle has a right angle.

   b) 4 m, 5 m, and 6 m

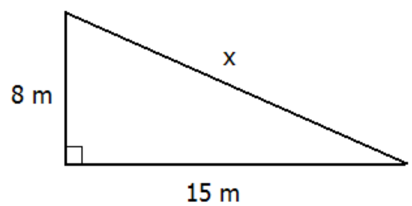
   Solution: The longest side is 6 m long. Thus, only this side can be the hypotenuse. We use the Pythagorean theorem to check for a right angle:

   $$4^2 + 5^2 \stackrel{?}{=} 6^2$$

   $$\begin{aligned} \text{LHS} &= 4^2 + 5^2 = 16 + 25 = 41 \\ \text{RHS} &= 6^2 = 36 \qquad \text{LHS} \neq \text{RHS} \end{aligned}$$

   We get that the two quantities are not equal, thus this triangle does not have a right angle.

2. Find the hypotenuse of the triangle shown on the figure.



   Solution: We apply the Pythagorean theorem. The longest side is always the one opposite the right angle.

   $$\begin{aligned} 8^2 + 15^2 &= x^2 \\ 289 &= x^2 \\ \pm 17 &= x \end{aligned}$$

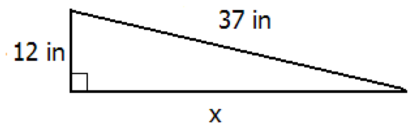
   Since distance can not be negative, -17 is ruled out. The answer is 17 m.

   Please note that the step taking us from $x^2 = 289$ to $x = \pm 17$ is a very nice shortcut. The tradiotional way of solving quadratic equations is to reduce one side to zero, factor, and apply the zero product rule.

   $$\begin{aligned} x^2 &= 289 \\ x^2 - 289 &= 0 \\ x^2 - 17^2 &= 0 \\ (x+17)(x-17) &= 0 \quad \Longrightarrow \quad x = -17 \text{ or } x = 17 \end{aligned}$$

   Students are encouraged to use the shorter version, **as long as they don't make the serious algebraic error** of concluding from $x^2 = 289$ that $x = 17$. While in the context of the geometry the negative solution is not possible, the equation $x^2 = 289$ has two solutions, 17 and -17.

3. Find the missing leg of the right triangle shown on the picture.



Solution: We apply the Pythagorean theorem. The longest side is always the one opposite the right angle.

$$\begin{aligned} (12\,\text{in})^2 + x^2 &= (37\,\text{in})^2 \\ x^2 + 144\,\text{in}^2 &= 1369\,\text{in}^2 && \text{subtract } 144\,\text{in}^2 \\ x^2 &= 1225\,\text{in}^2 && \sqrt{1225} = 35 \\ x &= \pm 35\,\text{in} \end{aligned}$$

Since distance can not be negative, $-35\,\text{in}$ is ruled out. The answer is 35 inches.

4. Find the distance between the points $(3,8)$ and $(8,-4)$.

Solution: We graph the points, and draw a horizontal and vertical line connecting the points as shown on the picture. We can compute the distance as the hypotenuse of the right triangle we created. How long are the shorter sides?

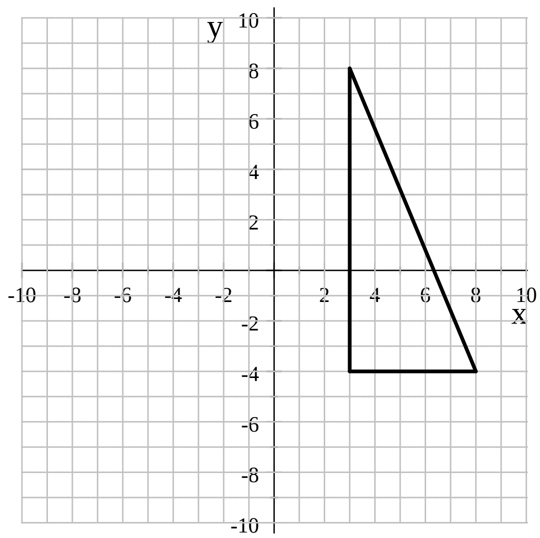
Algebraic approach: Subtract the coordinates. The length of the horizontal side is the difference between the $x-$coordinates: $8-3=5$ and the length of the vertical side is the difference between the $y-$coordinates: $8-(-4)=12$.

The difference will alwys work. Even if we get -5 instead of 5, it will not matter since we will square it in the Pythagorean theorem.

Geometric approach: From 3 to 8 we have to step 5 units up. From -4 to 8 : first we step 4 to get from -4 to 0. Then another 8 steps to 8, and so $4+8=12$ steps. The message here is that the algebra and geometry will always agree.

Now we know that the shorter sides are 5 and 12 units long, and we need to find the hypotenuse.

$$\begin{aligned} 5^2 + 12^2 &= x^2 \\ 25 + 144 &= x^2 \\ 169 &= x^2 \\ \pm 13 &= x \end{aligned}$$

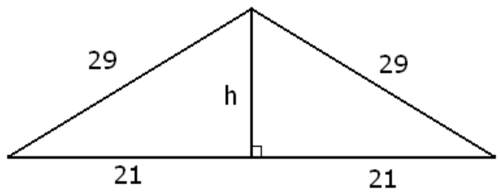


Since distances are never negative, -13 is ruled out and so the answer is 13 units.

5. The sides of an isosceles triangle are 42 units, 29 units, and 29 units long. Find the length of the height drawn to the 42 units long side.

   Solution: In case of isosceles triangles, the height drawn to the base splits the triangle into two identical right triangles as shown on the picture. The height now can be easily computed via the Pythagorean theorem.

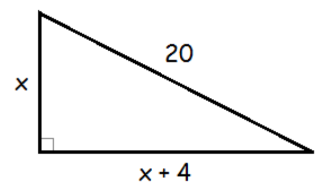
$$\begin{aligned} 21^2 + h^2 &= 29^2 \\ 441 + h^2 &= 841 \\ h^2 &= 400 \\ h &= \pm 20 \implies h = 20 \end{aligned}$$



   Again, the negative solution of the equation is ruled out because distances cannot be negative. The height belonging to the base is 20 units long.

6. The hypotenuse of a right triangle is $20\,\mathrm{cm}$. The difference between the other two sides is $4\,\mathrm{cm}$. Find the sides of the triangle.

   Solution: Let x denote the shortest side. Then the other missing side is $x + 4\,\mathrm{cm}$ long. We state the Pythagorean theorem for the triangle and solve the quadratic equation for x.



$$\begin{aligned} x^2 + (x+4)^2 &= 20^2 && \text{expand } (x+4)^2 \\ x^2 + x^2 + 8x + 16 &= 400 && \text{combine like terms} \\ 2x^2 + 8x + 16 &= 400 && \text{subtract 400} \\ 2x^2 + 8x - 384 &= 0 && \text{factor out 2} \\ 2\left(x^2 + 4x - 192\right) &= 0 \end{aligned}$$

   We will factor $x^2 + 4x - 192$ by trial and error. The negative sign in front of 192 indicates that one number is positive, the other is negative. Therefore, 4 is the difference of those two numbers. We list all pairs of factors for 192 until we find a pair with difference 4.

	192	
1	192	difference is 191
2	96	difference is 94
3	64	difference is 61
4	48	difference is 44
6	32	difference is 26
8	24	difference is 16
12	16	This is the one!

   Thus $x^2 + 4x - 192$ can be factored as $(x+16)(x-12)$. We apply the zero product rule.

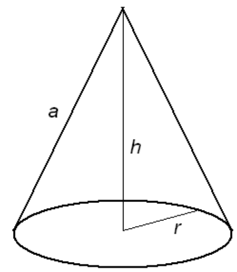
$$\begin{aligned} 2\left(x^2 + 4x - 192\right) &= 0 \\ 2(x+16)(x-12) &= 0 \end{aligned}$$

$$x_1 = -16, \; x_2 = 12$$

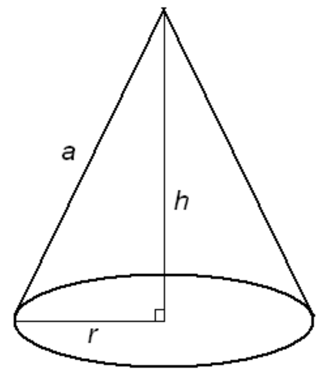
   Since distances are never negative, -16 is ruled out. If the shortest side is $12\,\mathrm{cm}$, the other side is $12\,\mathrm{cm} + 4\,\mathrm{cm} = 16\,\mathrm{cm}$. Thus the solution is $12\,\mathrm{cm}$ and $16\,\mathrm{cm}$. We check:

$$16 - 12 = 4 \checkmark \text{ and } 16^2 + 12^2 = 256 + 144 = 400 = 20^2 \checkmark$$

7. Find the height h of the cone shown on the picture, if the base has a radius of $10\,\mathrm{m}$ and $a = 26\,\mathrm{m}$.



Solution: There is a right triangle formed by a, h, and r as the picture shows.We state the Pythagorean theorem for this triangle and solve for h.



$$\begin{aligned} r^2 + h^2 &= a^2 \\ 10^2 + h^2 &= 26^2 \\ h^2 + 100 &= 676 \\ h^2 &= 576 \\ h &= \pm 24 \end{aligned}$$

The negative value is ruled out, and so the height is $24\,\mathrm{m}$.

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