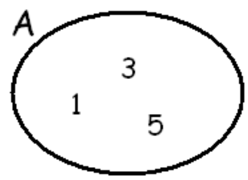
Definition: A **set** is a collection of objects. Two sets are equal if they contain the same objects.

Sets are usually denoted by uppercase letters. There are several ways a set could be given. We can describe a set using English language. In case of small sets, we can also simply list its elements separated by commas and enclosed in braces { }.

Example 1. Let M be the set of all one-digit natural numbers. Re-write this set by listing its elements.

We use the braces and list all one-digit natural numbers. $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

We can also describe a set using a diagram.
We depicted $A = \{1, 3, 5\}$.



Some famous sets have their own set theory label. For example, we already know the infinite set $\{1, 2, 3, 4, ...\}$. This is the set of all natural numbers or counting numbers, and it is denoted by $\mathbb{N}$.

Sometimes we need to be more descriptive when specifying sets.

Example 2. $S = \left\{x^2 : x \text{ is a natural number and } x \leq 5\right\}$.

We read this as S is a set containing x^2, where x is a natural number and x is less than or equal to 5. Of course, such a small set can be expressed much simpler, by listing its elements, as $S = \{1, 4, 9, 16, 25\}$. But this notation is often very useful when describing infinite sets.

Definition: A set is a collection of objects. The objects that make up the set are called the **elements** or **members** of the set, or that it **belongs** to the set.

Notation: If x is an element of a set S, we write by $x \in S$. If y is not an element of S, we write $y \notin S$.

Suppose that $A = \{1, 3, 5\}$. The following statements are all true.

$3 \in A$	read as: 3 belongs to A	$4 \notin A$	read as: 4 does not belong to A
$5 \in A$	read as: 5 is an element of A	$-6 \notin A$	read as: -6 is not an element of A

Example 3. Suppose that $A = \{1, 3, 5\}$ and that $\mathbb{N}$ is the set of all natural numbers, in short, $\mathbb{N} = \{1, 2, 3, 4, ...\}$. Determine whether the given statements are true or false.

a) $1 \in A$ b) $2 \in A$ c) $-1 \in \mathbb{N}$ d) $5 \notin \mathbb{N}$ e) $4 \notin A$

Solution: a) The statement $1 \in A$ reads: 1 *belongs to set* A. This is true as 1 is an element of set A.

b) The statement $2 \in A$ reads: 2 *belongs to set* A. This is not true as 2 is not an element of set A.

c) The statement $-1 \in \mathbb{N}$ reads: -1 *belongs to the set of all natural numbers*.
(In short, -1 is a natural number). This statement is false.

d) The statement $5 \notin \mathbb{N}$ reads: 5 *does not belong to the set of all natural numbers*.
(In short, 5 is not a natural number). This statement is false.

e) The statement $4 \notin A$ reads: 4 *does not belong to set* A. This statement is true.

Definition: Two sets are **equal** if they contain the same objects. We use the symbol $=$ to denote equal sets.

When writing a set, the order of listing and repetition of its elements does not change a set.

Example 4. Let A be the set of odd natural numbers between 0 and 6. Suppose further that $B = \{5, 1, 3\}$, and that $C = \{1, 5, 1, 5, 1, 1, 1, 3, 3\}$. Then all three sets are equal to each other, i.e $A = B = C$.

Since we are free to list the elements of a set any way we want to, it is often startegic to keep things organized by listing elements in increasing order. In this case, $A = B = C = \{1, 3, 5\}$. Sometimes we have reasons to part from this convention.

Definition: The **set of all integers**, denoted by $\mathbb{Z}$, is the set containing all natural numbers, their opposites, and zero.

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots\}$$

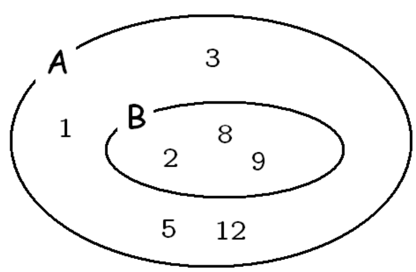
Some people prefer to present the set of all integers sort of organized, as $\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$. The disadvantage here is that both the beginning and the end of this infinite list goes on forever. More precisely, there is no beginning and no end. Both presentations are commonly used.

Definition: There is a unique set that contains no elements. It is called the **empty set** and is denoted by $\varnothing$ or by $\{\quad\}$.

Definition: Set B is a **subset** of set A if all elements of B also belong to A. Notation: $B \subseteq A$

There is another way to express this relationship. B is a subset of A if for all things x in the world, if x is an element of B, then x is also an element of A. This approach might be helpful later.

Example 5. Suppose that $A = \{1, 2, 3, 5, 8, 9, 12\}$ and $B = \{2, 8, 9\}$. Then B is a subset of A.



Example 6. Suppose that $X = \{a, b, d, f\}$ and $Y = \{a, b, c, d, e, f, g\}$. Then $X \subseteq Y$.

Example 7. Suppose that $S = \{1, 4, 9, 16\}$. Then $S \subseteq S$.

While this might look strange at first, the definition of subset applies. Every element of S is an element of S. Perhaps this statement is similar to $5 \leq 5$. For every number x, the statement $x \leq x$ is true.

Even more interestingly, the empty set is also a subset of every set. This is because the definition applies, even if strangely so. For every object x in the world, if x is in the empty set (it's not), then it in set A. We say that this statement is **vacuously true**.

Theorem: For all sets S, the following are both true: $\varnothing \subseteq S$ and $S \subseteq S$.

Example 8. If $\mathbb{N}$ is the set of all natural numbers and $\mathbb{Z}$ is the set of all integers, then $\mathbb{N} \subseteq \mathbb{Z}$.

Example 9. Suppose that E is the set of all even natural numbers, $E = \{2, 4, 6, 8, 10, \ldots\}$, and recall that $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. Then $E \subseteq \mathbb{N}$.

Example 10. Suppose that L is the set of all letters in the English alphabet, and V is the set of vowels in the English alphabet. Then $V \subseteq L$.

Example 11. Let M be the set of all mammals and D the set of all dogs. Then D is a subset of M, or, in short, $D \subseteq M$.

Example 12. List all subsets of the given set if

a) $A = \{1\}$ b) $B = \{1, 2\}$ c) $C = \{1, 2, 3\}$

Solution: a) The set $\{1\}$ has two subsets: $\varnothing$ and $\{1\}$.

b) The set $\{1, 2\}$ has four subsets: $\varnothing$, $\{1\}$, $\{2\}$, and $\{1, 2\}$.

c) As the given sets become larger, it becomes important to be systematic when finding the subsets. Let us list the subsets of C by organizing by the number of its elements.

0-element subsets:	$\varnothing$	
1-element subsets:	$\{1\}, \{2\}, \{3\}$	
2-element subsets:	$\{1, 2\}, \{1, 3\}, \{2, 3\}$	
3-element subsets:	$\{1, 2, 3\}$	So there are 8 subsets.

Practice Problems

1. Suppose that S is a set defined as $S = \{-2, 4, 5, 16\}$ and recall that $\mathbb{N} = \{1, 2, 3, 4, ...\}$. Determine whether the given statements are true or false.

   a) $-2 \in S$ b) $-2 \in \mathbb{N}$ c) $-3 \notin \mathbb{N}$ d) $5 \notin S$ e) $1 \in \mathbb{N}$

2. Let $A = \{1, 2, 5, 8, 9\}$ and $B = \{2, 4, 6, 8\}$. Draw a Venn diagram depicting these sets.

3. Suppose that $P = \{1, 7, 8\}$ and $Q = \{1, 2, 5, 7, 8\}$. Label each of the following statements as true or false.

   a) $P \subseteq \mathbb{Z}$ b) $Q \subseteq P$ c) $P \subseteq Q$ d) $\mathbb{N} \subseteq Q$ e) $\varnothing \subseteq P$

4. Find each of the following sets and if possible, present them by listing their elements.

   a) $S = \{x : x$ is a natural number such that $x < 4$ and $x < 7\}$

   b) $A = \{a : a$ is a natural number such that $a < 4$ or $a < 7\}$

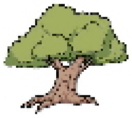
   c) $P = \{y : y$ is an integer such that $y > 3$ and $y < 8\}$

   d) $M = \{y : y$ is an integer such that $y > 3$ or $y < 8\}$

5. Suppose that $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 3, 4\}$. Explain why $Y \in X$ is a false statement.

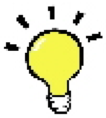
6. Find two sets A and B so that both $A \subseteq B$ and $B \subseteq A$ are true.

7. List all subsets of $A = \{1, 2, 3, 4\}$.

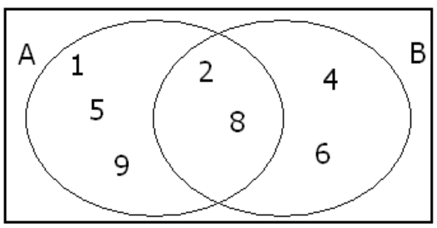


Enrichment

1. Suppose that A and B are sets such that $A \cap B = \{1, 2, 5\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$. How many different sets are possible for A?
2. Our junior class had 60 students. If 42 students took history, 35 students took French, and 19 took both history and French, how many students in the junior class took neither French nor history? (Hint: this is the kind of problem in which a Venn diagram can be very helpful.)



Answers

1. a) true b) false c) true d) false e) true
2. 

3. a) true b) false c) true d) false e) true
4. a) $\{1, 2\}$ b) $\{1, 2, 3, 4, 5, 6\}$ c) $\{3, 4, 5, 6, 7\}$ d) $\mathbb{Z}$ (the set of all integers)
5. Y is not an element of set X. Instead, Y is a subset of X, denoted by $Y \subseteq X$.
6. This is naturally true if $A = B$ as every set is a subset of itself. However, if A and B are different sets, at least one of $A \subseteq B$ and $B \subseteq A$ will be false.
7. We will list the subsets of A by organizing by the number of their elements.

0-element subsets:	$\varnothing$	1 subset
1-element subsets:	$\{1\}, \{2\}, \{3\}, \{4\}$	4 subsets
2-element subsets:	$\{1, 2\}$ $\{1, 3\}$ $\{2, 3\}$ $\{1, 4\}$ $\{2, 4\}$ $\{3, 4\}$	6 subsets
3-element subsets:	$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$	4 subsets
4-element subsets:	$\{1, 2, 3, 4\}$	1 subset

So there are 16 subsets.

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