We have previously studied sets. At this point, we can define and compare sets. We will now start studying operations on sets.

**Definition:** If $A$ and $B$ are sets, then the **intersection** of $A$ and $B$, denoted by $A \cap B$, is the set such that for all $x$,

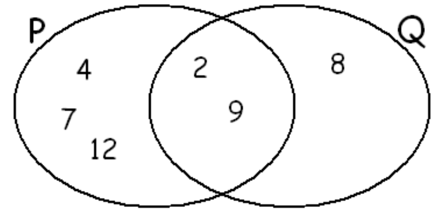
$x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

The intersection of two sets is the set of all elements that belong to *both* sets.

**Example 1.** Suppose that $P = \{2, 4, 7, 9, 12\}$ and $Q = \{2, 8, 9\}$. Find $P \cap Q$.

**Solution:** The intersection of $P$ and $Q$ is the set containing those elements that are in **both** $P$ and $Q$. Since $P$ and $Q$ are small sets, we check from element to element, and collect those that belong to both. We can see that $P \cap Q = \{2, 9\}$.

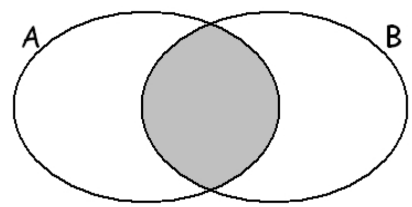
A picture such as this one is called a Venn diagram. Venn diagrams often provide useful visual tools to solve set theory problems. We can depict the intersection using Venn Diagrams.




**Example 2.** Suppose that $A = \{1, 3, 5, 7, 9\}$ and $B = \{5, 6, 7, 8, 9, 10\}$. Find $A \cap B$.

**Solution:** The intersection of $A$ and $B$ is the set containing those elements that are in both $A$ and $B$. Since $A$ and $B$ are small sets, we check from element to element, and collect those that belong to both. We can see that $A \cap B = \{5, 7, 9\}$.

If we use a Venn diagram, the intersection of the two sets is the 'overlap' between the two sets as shown.




The shaded region is $A \cap B$

**Example 3.** Suppose that $T = \{3, 4, 7, 10\}$ and $Q = \{1, 6, 8\}$. Find $T \cap Q$.

**Solution:** As we look for elements in common, we find none. Thus $T \cap Q = \varnothing$. When this happens, we say that the two sets are **disjoint**.

We want the intersection of two sets to always be a set. In other words, we want the set of sets (!) to be closed under intersection. This is why it was important for us to define $\varnothing$, the empty set.

**Example 4.** Find $\mathbb{N} \cap \mathbb{Z}$.

**Solution:** If we start with the natural numbers, we notice that they are automatically in $\mathbb{Z}$. Indeed, $\mathbb{N}$ is a subset of $\mathbb{Z}$, and so every element in $\mathbb{N}$ is also in $\mathbb{Z}$ and thus in both sets. However, with the negative integers and zero we find that they are not in both sets because they are not in $\mathbb{N}$. Thus the intersection of the two sets is $\mathbb{N}$. In short, $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}$.

**Example 5.** Let $S = \{3, 8, 14\}$. Find $S \cap \varnothing$.

**Solution:** The intersection of two sets is the set of elements in *both* sets. Since there is nothing in the empty set, there cannot be anything in the intersection. Thus $S \cap \varnothing = \varnothing$

**Definition:** If $A$ and $B$ are sets, then the **union** of $A$ and $B$, denoted by $A \cup B$, is the set such that for all $x$,

$x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

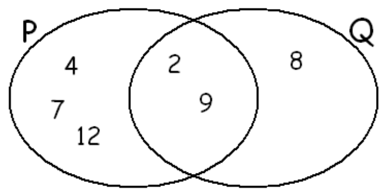
The word 'or' is used in the strict mathematical sense. $x \in A$ or $x \in B$ is true if either $x$ is in $A$ only, or $x$ is in $B$ only, or if $x$ is in both. So, $x$ is in the union of $A$ and $B$ if it is in $A$, in $B$, or in both $A$ and $B$.

The union of two sets is the set of all elements from one set, put together with the set of all elements of the other. Imagine we throw the elements of both sets together and then we list them as elements of a single set, ignoring repetitions.

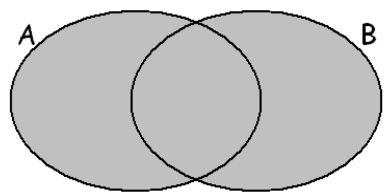
**Example 6.** Suppose that $P = \{2, 4, 7, 9, 12\}$ and $Q = \{2, 8, 9\}$. Find $P \cup Q$.

**Solution:** The union of $P$ and $Q$ is the set containing those elements that are in $P$ or in $Q$. Since $P$ and $Q$ are small sets, we check from element to element, and collect those that belong to either sets or to both. Another way of visualizing the union is to throw together $P$ and $Q$ and list the resulting set without repetition. We can see that $P \cup Q = \{2, 4, 7, 8, 9, 12\}$.

A Venn diagram might help again. For the union, we collect every element from the three separate regions.




If we use a Venn diagram, the union of the two sets is the collection of those three regions as shown.




The shaded region is $A \cup B$

**Example 7.** Find $\mathbb{N} \cup \mathbb{Z}$.

**Solution:** Let us start with the integers this time. All integers are in the union since they are in $\mathbb{Z}$. Now we look at the other set, $\mathbb{N}$, and notice that all natural numbers are already listed in the union because they are automatically in $\mathbb{Z}$. Indeed, $\mathbb{N}$ is a subset of $\mathbb{Z}$, and so every element in $\mathbb{N}$ is also in $\mathbb{Z}$. Therefore, $\mathbb{N}$ does not bring anything new to the union, and so $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$.

**Example 8.** Suppose $T = \{2, 10, 12, 30\}$. Find $T \cup \varnothing$

**Solution:** The union obviously contains all four elements of $T$. Now we need to add the elements that are not in $T$ but are in the empty set. Since there isn't anything in the empty set, it does not bring anything new to the union, and so $T \cup \varnothing = \{2, 10, 12, 30\}$. We can also state the answer as $T \cup \varnothing = T$.

# Practice Problems

1. Suppose that $P = \{1, 4, 6, 9\}$ and $Q = \{1, 2, 3, 4, 5\}$. Find each of the following.

   a) $P \cap Q$   b) $P \cup Q$   c) $P \cap \varnothing$   d) $Q \cup \varnothing$

2. Let $A = \{1, 2, 5, 8, 9\}$ and $B = \{2, 4, 6, 8\}$.

   a) Draw a Venn diagram depicting these sets.

   b) Find each of the following.

   i) $A \cap B$   ii) $A \cup B$   iii) $B \cup (A \cap B)$

   c) Label each of the following statements as true or false.

   i) $A \subseteq A \cap B$   ii) $B \subseteq A \cup B$   iii) $A \cap B \subseteq A \cup B$

3. Let $P$ denote the set of all students taking physics at Truman College. Let $M$ denote the set of all students taking mathematics at Truman College.

   a) describe the set $P \cap M$   b) describe the set $P \cup M$

4. Label each of the following statements as true or false.

   a) $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}$   b) $\mathbb{N} \cap \mathbb{Z} = \mathbb{Z}$   c) $\mathbb{N} \cup \mathbb{Z} = \mathbb{N}$   d) $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$

5. Label each of the following statements as true or false. (Hint: make up suitable examples for yourself and then investigate!)

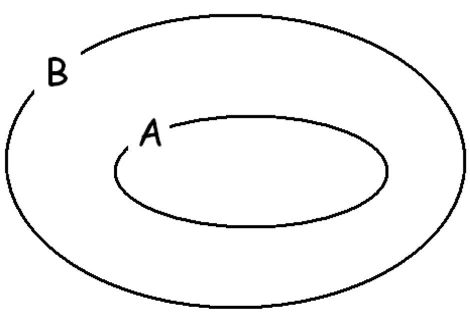
   a) If $A \subseteq B$, then $A \cap B = A$

   b) If $A \subseteq B$, then $A \cup B = B$

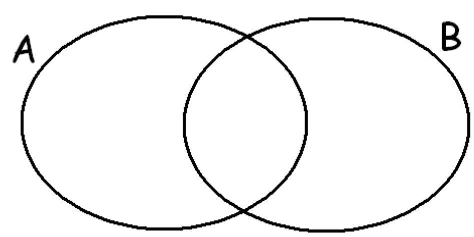
   c) For all sets $A$ and $B$, $A \cap B \subseteq A$

   d) For all sets $A$ and $B$, $B \subseteq A \cup B$

6. Recall that a visual representation of subset is to draw one set inside the other. However, this is not a Venn diagram. In case of a Venn diagram, we must have the three distinct regions.




This is not a Venn diagram.




This is a Venn diagram.

   Given a Venn diagram depicting sets $A$ and $B$, how does it show up that $A$ is a subset of $B$?

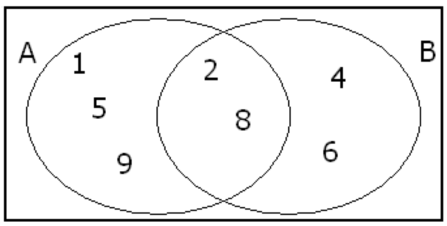
# Enrichment

Suppose that $A$ and $B$ are sets such that $A \cap B = \{1, 2, 5\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$. How many different sets are possible for $A$?

# Answers

1. a) $\{1, 4\}$ b) $\{1, 2, 3, 4, 5, 6, 9\}$ c) $\varnothing$ d) $\{1, 2, 3, 4, 5\}$ or $Q$

2. a)




b) i) $\{2, 8\}$ ii) $\{1, 2, 4, 5, 6, 8, 9\}$ iii) $\{2, 4, 6, 8\}$

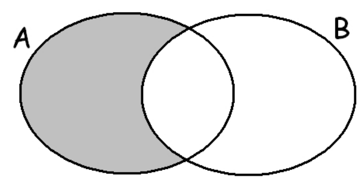
c) i) false ii) true iii) true

3. a) $P \cap M$ - the set of all students taking mathematics and physics at Truman College.
   b) $P \cup M$ - the set of all students taking mathematics or physics or both at Truman College.

4. a) true b) false c) false d) true

5. a) true b) true c) true d) true

6. If $A$ is a subset of $B$, then there is no element that belongs to $A$ but not to $B$. This will show up on the Venn diagram by the shaded region shown containing no elements.

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