In the previous section, we learned how to add and subtract algebraic expressions. In this handout, we will multiply them.

> To multiply an algebraic expression by a number or a one-term expression, we apply the distributive law.

Example 1. Expand the products as indicated.

a) $3x\left(5x^2 - x + 1\right)$ c) $5ax\,(2a - x)$ e) $-3ab^3\,(4a - b + 2ab)$

b) $-1\left(-x^2 + 3x - 4\right)$ d) $-ab\,(3a - 5b - 1)$

Solution: a) $3x\left(5x^2 - x + 1\right) = \boxed{15x^3 - 3x^2 + 3x}$

b) $-1\left(-x^2 + 3x - 4\right) = \boxed{x^2 - 3x + 4}$

Note that multipliction by -1 means we change the sign in front of each term.

c) $5ax\,(2a - x) = 5ax \cdot 2a - 5ax \cdot x = \boxed{10a^2x - 5ax^2}$

d) $-ab\,(3a - 5b - 1) = -ab \cdot 3a - ab\,(-5b) - ab\,(-1) = \boxed{-3a^2b + 5ab^2 + ab}$

e) $-3ab^3\,(4a - b + 2ab) = -3ab^3\,(4a) + \left(-3ab^3\right)(-b) + \left(-3ab^3\right)(2ab) = \boxed{-12a^2b^3 + 3ab^4 - 6a^2b^4}$

When multiplying algebraic expressions, we apply the distributive law and then combine like terms.

Example 2. Expand each of the following.

a) $(x - 3)(-2x + 5)$ b) $(3a - 2b)(5x - y)$ c) $(2y + 1)(3y - 5)$

Solution: When we multiply two two-term expressions, FOIL is just one of the possible ways to make sure we applied the distributive law. (F - first with first; O - the outer terms; I - the inner terms; L - last with last)

a) $(x - 3)(-2x + 5) = \overbrace{x \cdot (-2x)}^{\text{F}} + \overbrace{x \cdot 5}^{\text{O}} + \overbrace{(-3) \cdot (-2x)}^{\text{I}} + \overbrace{(-3) \cdot 5}^{\text{L}}$

$= -2x^2 + 5x + 6x - 15$ $5x$ and $6x$ are like terms

$= \boxed{-2x^2 + 11x - 15}$

b) $(3a - 2b)(5x - y) = \overbrace{3a \cdot 5x}^{\text{F}} + \overbrace{3a \cdot (-y)}^{\text{O}} + \overbrace{(-2b) \cdot (5x)}^{\text{I}} + \overbrace{(-2b) \cdot (-y)}^{\text{L}}$

$= 15ax - 3ay - 10bx + 2by$ all terms are unlike

$= \boxed{15ax - 3ay - 10bx + 2by}$

c) $(2y + 1)(3y - 5) = \overbrace{2y \cdot 3y}^{\text{F}} + \overbrace{2y \cdot (-5)}^{\text{O}} + \overbrace{1 \cdot 3y}^{\text{I}} + \overbrace{1 \cdot (-5)}^{\text{L}}$

$= 6y^2 - 10y + 3y - 5$ $-10y$ and $3y$ are like terms

$= \boxed{6y^2 - 7y - 5}$

If the multiplication is more complicated, the distributive law is applied so that each term from one expression is multiplied by each term of the other expression exactly once.

Example 3. Expand each of the following.

a) $(x+2)\left(3x^2-5x-7\right)$ b) $(2a-b-1)(5a-4b)$

Solution: a) We will first multiply all three terms of the second expression by x, and then by 2. We expect six little products. Finally, we combine like terms.

$$(x+2)\left(3x^2-5x-7\right) = x\cdot 3x^2 + x\cdot(-5)x + x\cdot(-7) + 2\cdot 3x^2 + 2\cdot(-5x) + 2\cdot(-7)$$

$$= 3x^3 - 5x^2 - 7x + 6x^2 - 10x - 14 = \boxed{3x^3 + x^2 - 17x - 14}$$

b) $(2a-b-1)(5a-4b) = 2a\cdot 5a + 2a\cdot(-4b) + (-b)\,5a + (-b)(-4b) + (-1)\cdot 5a + (-1)(-4b)$

$= 10a^2 - 8ab - 5ab + 4b^2 - 5a + 4b$ $\qquad$ $-8ab$ and $-5ab$ are like terms.

$= \boxed{10a^2 - 13ab + 4b^2 - 5a + 4}$

Example 4. Expand each of the following.

a) $(x+1)^2$ c) $(2y-1)^2$ e) $(3x-1)(3x+1)$

b) $(3a-2)^2$ d) $(2y-1)^3$ f) $(a+b)(a-b)$

Solution: When we square an entire sum or difference, the expression is called a **complete square**.

a) $(x+1)^2 = (x+1)(x+1) = x\cdot x + x\cdot 1 + 1\cdot x + 1\cdot 1 = x^2 + x + x + 1 = \boxed{x^2 + 2x + 1}$

b) $(3a-2)^2 = (3a-2)(3a-2) = 9a^2 - 6a - 6a + 4 = \boxed{9a^2 - 12a + 4}$

c) $(2y-1)^2 = (2y-1)(2y-1) = 4y^2 - 2y - 2y + 1 = \boxed{4y^2 - 4y + 1}$

d) In solving this problem, we will apply our result from the previous problem.

$$(2y-1)^3 = (2y-1)(2y-1)(2y-1) = (2y-1)(2y-1)^2 = (2y-1)\left(4y^2 - 4y + 1\right)$$

$$= 8y^3 - 8y^2 + 2y - 4y^2 + 4y - 1 = \boxed{8y^3 - 12y^2 + 6y - 1}$$

e) The expressions $3x-1$ and $3x+1$ are **conjugates** of each other. Conjugates have very useful properties.

$(3x-1)(3x+1) = 9x^2 + 3x - 3x - 1 = \boxed{9x^2 - 1}$

f) $(a+b)(a-b) = a^2 - ab + ba - b^2 = a^2 - ab + ab - b^2 = \boxed{a^2 - b^2}$

Discussion: Explain why re-writing $2(x-3)^2$ as $(2x-6)^2$ would be an incorrect step.

Example 5. Simplify the expression $(2x-3)^2 - (2x+1)(3x-7)$

Solution: If we consider the expression as one that connects algebraic expressions, then there are three operations: squaring, a multiplication, and a subtraction. We apply order of operations. We will work out the products separatly.

Exponentiation first: $(2x-3)^2 = (2x-3)(2x-3) = 4x^2 - 6x - 6x + 9 = 4x^2 - 12x + 9$

then multiplication: $(2x+1)(3x-7) = 6x^2 - 14x + 3x - 7 = 6x^2 - 11x - 7$

and finally subtraction.

$(2x-3)^2 - (2x+1)(3x-7) = \left(4x^2 - 12x + 9\right) - \left(6x^2 - 11x - 7\right)$ $\qquad$ to subtract is to add the opposite

$= 4x^2 - 12x + 9 - 6x^2 + 11x + 7 = \boxed{-2x^2 - x + 16}$

Caution! This last example contains a situation in which students are at risk to make an error. After we expand an expression such as $(2x+1)(3x-7)$, we usually do not need a parentheses. However, in this case, the product is subtracted and so we still need a pair of parentheses indicating that we are subtracting the entire expression and not just its first term. This is just one of the several examples in which we are tempted to violate the distributive law.

Example 6. Solve the given equation. Make sure to check your solutions.

$$(2x-3)^2-(x+1)(3x-5)=11-(x-1)(3-x)$$

Solution: We carefully expand the indicated products and combine like terms. Notice that even after we expanded $(x+1)(3x-5)$ and $(x-1)(3-x)$, we still need to keep them in parentheses because we are subtracting them. We will first work out the products.

$$(2x-3)^2=(2x-3)(2x-3)=4x^2-6x-6x+9=4x^2-12x+9$$

$$(x+1)(3x-5)=3x^2-5x+3x-5=3x^2-2x-5$$

$$(x-1)(3-x)=3x-x^2-3+x=-x^2+4x-3$$

We are now ready to begin to solve the equation.

$$\begin{aligned}
(2x-3)^2-(x+1)(3x-5) &= 11-(x-1)(3-x) && \\
4x^2-12x+9-\left(3x^2-2x-5\right) &= 11-\left(-x^2+4x-3\right) && \text{to subtract is to add the opposite}\\
4x^2-12x+9-3x^2+2x+5 &= 11+x^2-4x+3 && \text{combine like terms}\\
x^2-10x+14 &= x^2-4x+14 && \text{subtract } x^2\\
-10x+14 &= -4x+14 && \text{add } 10x\\
14 &= 6x+14 && \text{subtract } 14\\
0 &= 6x && \text{divide by } 6\\
0 &= x &&
\end{aligned}$$

We check: if $x=0$, then

$$\text{LHS}=(2\cdot 0-3)^2-(0+1)(3\cdot 0-5)=(-3)^2-1(-5)=9+5=14$$

$$\text{RHS}=11-(0-1)(3-0)=11-(-1)3=11+3=14$$

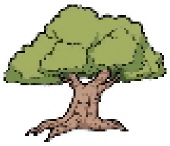
and so our solution, $\boxed{x=0}$ is correct.

Example 7. If we increase the length of each side of a square by $4\,\text{cm}$, the area of the square increases by $64\,\text{cm}^2$. How long are the sides before the increase?

Solution: Let us denote the length of the original square by x. Then its area is x^2. The side of the larger square is $x+4$. Therefore, the area of the larger square is $(x+4)^2$. The equation will express the comparison between the two areas.

$$\begin{aligned}
(x+4)^2 &= x^2+64 && \text{expand complete square on the left-hand side}\\
x^2+8x+16 &= x^2+64 && \text{subtract } x^2\\
8x+16 &= 64 && \text{subtract } 16\\
8x &= 48 && \text{divide by } 8\\
x &= 6 &&
\end{aligned}$$

Thus the original square has sides $6\,\text{cm}$ long. The area is $36\,\text{cm}^2$. If we increased each side by $4\,\text{cm}$, the new side is $10\,\text{cm}$, and the new area $100\,\text{cm}^2$. Indeed, the two areas differ by $100\,\text{cm}^2-36\,\text{cm}^2=64\,\text{cm}^2$. Thus our solution is correct: the original square has sides that are $\boxed{6\,\text{cm}}$ long.



Enrichment

1. Special products. Expand each of the following.

   a) $(a+b)^2$ c) $(a+b)^3$ e) $(x-y)(x+y)$ g) $(x-y)\left(x^3+x^2y+xy^2+y^3\right)$

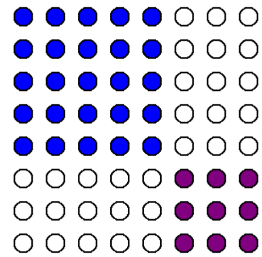
   b) $(a-b)^2$ d) $(a-b)^3$ f) $(x-y)\left(x^2+xy+y^2\right)$

2. Explain how $2x+3x=5x$ can be explained in terms of the distributive law.

3. The *Freshman's Dream Error*. We saw that

   $$3^2+5^2=9+25=34 \text{ and } (3+5)^2=8^2=64.$$

   To confuse a^2+b^2 - a sum of two squares with the complete square $(a+b)^2$ is an error so egregious, it actually has a name: The Freshman's Dream. The Freshman's Dream is a violation of the distributive law in which we state that $(a+b)^2=a^2+b^2$. Use the picture given to illustrate the error and the correct formula for the expansion of $(a+b)^2$.



Sample Problems

1. Multiply the algebraic expressions as indicated.

   a) $3xy\left(2x^2+4y-5\right)$ b) $-5x^3\left(2x^2-x+8\right)$ c) $-\left(-x+3y-8z^2+6\right)$ d) $-2a\left(-a+3b^2-2ab+7\right)$

2. Multiply the algebraic expressions as indicated.

   a) $(x+3)(5x-3)$ b) $(5-2x)^2$ c) $(x+4)(1-2x)$ d) $-2(x-3)^2$

3. Simplify each of the following expressions.

   a) $(x-5)^2-(2x-1)(x+3)$ b) $-(m-3)^2$ c) $-2(3x-5)-(2x-1)^2$

4. Solve each of the following equations. Make sure to check your solutions.

   a) $(x-3)^2-(2x-5)(x+1)=5-(x-1)^2$ c) $12-(2p-1)(p+1)=-2(-p+5)^2$

   b) $(x+1)^2-(2x-1)^2+(3x)^2=6x(x-2)$

5. If we increase each side of a square by $3\,\mathrm{cm}$, its area increases by $51\,\mathrm{cm}^2$. How long are the sides before the increase?

Practice Problems

1. Multiply the algebraic expressions as indicated.

   a) $5a^2b\left(-3ab^2+y-5\right)$ b) $0\left(2x^2-7x+8\right)$ c) $-16t^3\left(-t^2+6t-1\right)$ d) $-(x-5a+3ax-1)$

2. Multiply the algebraic expressions as indicated.

   a) $(3-2x)(x-7)$ b) $(3-2x)(3x-2)$ c) $(3-2x)(3+2x)$ d) $(3-2x)^2$ e) $2(a-1)^2$

3. Simplify each of the following. Notice what is the same and what is different in the problems.

   a) $(-2x+5)+(3x-8)$ b) $(-2x+5)-(3x-8)$ c) $3(-2x+5)-2(3x-8)$ d) $(-2x+5)(3x-8)$

4. Simplify each of the following expressions.

   a) $(x+2)^2-(x-2)^2$ b) $1-(x-4)(3x-1)$ c) $-3(5m-1)^2$ d) $3x-1-3x(3x-1)$

5. Solve each of the following equations.

   a) $2x(3x-1)-x(5x-1)=(x-1)^2$
   b) $y^2-(y-1)^2+(y-2)^2=(y-3)(y-5)$
   c) $(3x)^2-(x+3)(5x-3)=(5-2x)^2-16$
   d) $(w+4)(1-2w)=3w-2(w-3)^2$
   e) $(2x-3)^2-3(x-2)^2=10-(x-2)(7-x)$
   f) $(2-w)^2-(2w-3)^2+7=(w-2)(5-3w)$
   g) $3(a+11)-a(8-3a)=3(a-2)^2$
   h) $-5(2x-1)-(4-x)^2=3-(x+1)^2$
   i) $5(-3-x)-3x(x-2)=x-3(x+2)(x-5)$
   j) $2(-m-2)^2-(m-2)^2=8m+(m+2)^2$
   k) $(3a-5)(2-a)-(2a-1)(a+3)=-5a^2-7$

6. If we increase the length of each side of a square by $2\,\mathrm{cm}$, the area of the square increases by $24\,\mathrm{cm}^2$. How long are the sides before the increase?

Answers

Sample Problems

1. a) $6x^3y + 12xy^2 - 15xy$ b) $-10x^5 + 5x^4 - 40x^3$ c) $x - 3y + 8z^2 - 6$ d) $2a^2 - 6ab^2 + 4a^2b - 14a$
2. a) $5x^2 + 12x - 9$ b) $4x^2 - 20x + 25$ c) $-2x^2 - 7x + 4$ d) $-2x^2 + 12x - 18$
3. a) $-x^2 - 15x + 28$ b) $-m^2 + 6m - 9$ c) $-4x^2 - 2x + 9$
4. 2 5. 0 6. 3 7. 7 cm

Practice Problems

1. a) $-15a^3b^3 + 5a^2by - 25a^2b$ b) 0 c) $16t^5 - 96t^4 + 16t^3$ d) $-x + 5a - 3ax + 1$
2. a) $-2x^2 + 17x - 21$ b) $-6x^2 + 13x - 6$ c) $9 - 4x^2$ d) $4x^2 - 12x + 9$ e) $2a^2 - 4a + 2$
3. a) $x - 3$ b) $-5x + 13$ c) $-12x + 31$ d) $-6x^2 + 31x - 40$
4. a) $8x$ b) $-3x^2 + 13x - 3$ c) $-75m^2 + 30m - 3$ d) $-9x^2 + 6x - 1$
5. a) 1 b) 2 c) 0 d) 1 e) 3 f) 4 g) -3 h) no solution i) -5 j) all real numbers k) 0
6. 5 cm

Answers - Enrichment

1. a) $(a+b)^2 = a^2 + 2ab + b^2$ b) $(a-b)^2 = a^2 - 2ab + b^2$ c) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
   d) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ e) $(x-y)(x+y) = x^2 - y^2$
   f) $(x-y)\left(x^2 + xy + y^2\right) = x^3 - y^3$ g) $(x-y)\left(x^3 + x^2y + xy^2 + y^3\right) = x^4 - y^4$
2. $5x = x \cdot 5 = x(2+3) = 2x + 3x$
3. Let $a = 5$ and $b = 3$. The expression $a^2 + b^2$ or $5^2 + 3^2$ is the combined region of the blue square and the purple square.

   On the other hand, $(5+3)^2$ can be seen as the entire region, the 8 by 8 square. Clearly $34 \neq 64$.

   $(a+b)^2$ is the entire image (Still $a = 5$ and $b = 3$)
   a^2 is the blue region, b^2 is the purple region. The white rectangular dots represent ab and ba

   So, $(a+b)^2 = a^2 + b^2 + 2ab$

Solutions - Sample Problems

1. Multiply the algebraic expressions as indicated.

a) $3xy\left(2x^2+4y-5\right)$

Solution: We distribute $3xy$. $3xy\left(2x^2+4y-5\right)=\boxed{6x^3y+12xy^2-15xy}$

b) $-5x^3\left(2x^2-x+8\right)$

Solution: We distribute $-5x^3$. $-5x^3\left(2x^2-x+8\right)=\boxed{-10x^5+5x^4-40x^3}$

c) $-\left(-x+3y-8z^2+6\right)$

Solution: The notation here indicates multiplication by -1, which is the same as taking the opposite of a quantity. We distribute -1.

$$-1\left(-x+3y-8z^2+6\right)=\boxed{x-3y+8z^2-6}$$

d) $-2a\left(-a+3b^2-2ab+7\right)$

Solution:

$$-2a\left(-a+3b^2-2ab+7\right)=\boxed{2a^2-6ab^2+4a^2b-14a}$$

2. Multiply the algebraic expressions as indicated.

a) $(x+3)(5x-3)$

Solution: We expand the expression using the distributive law. In the simplest case, when both expressions have only two terms, we use FOIL (F - first term with first term, O - outer terms, I - inner terms, L - last terms)

$$\begin{aligned}(x+3)(5x-3) &= && \text{FOIL}\\ 5x^2-3x+15x-9 &= && \text{combine like terms}\\ &= \boxed{5x^2+12x-9}\end{aligned}$$

b) $(5-2x)^2$

Solution: To square something means to write it down twice and multiply.

$$\begin{aligned}(5-2x)^2 &= (5-2x)(5-2x) && \text{FOIL}\\ &= 25-10x-10x+4x^2 && \text{combine like terms}\\ &= \boxed{4x^2-20x+25}\end{aligned}$$

c) $(x+4)(1-2x)$

Solution: We expand the expression and combine like terms.

$$(x+4)(1-2x)=x-2x^2+4-8x=\boxed{-2x^2-7x+4}$$

d) $-2(x-3)^2$

Solution: We have two operations: multiplication by -2 and exponentiation. Order of operations still apply, thus we start with exponentiation.

$$-2(x-3)^2=-2((x-3)(x-3))=-2\left(x^2-3x-3x+9\right)=-2\left(x^2-6x+9\right)=\boxed{-2x^2+12x-18}$$

3. Simplify each of the following expressions.

a) $(x-5)^2-(2x-1)(x+3)$

Solution: If we consider this as operations on algebraic expressions, then we are faced with an exponentiation, a multiplication, and a subtraction. We will execute them exactly in this order.

$$(x-5)^2=(x-5)(x-5)=x^2-5x-5x+25=x^2-10x+25$$

$$(2x-1)(x+3)=2x^2+6x-x-3=2x^2+5x-3$$

$$(x-5)^2-(2x-1)(x+3)=x^2-10x+25-\left(2x^2+5x-3\right)=$$

$$=x^2-10x+25+\left(-2x^2-5x+3\right)=\boxed{-x^2-15x+28}$$

b) $-(m-3)^2$

Solution: We are asked to take the opposite of a complete square.

$$-(m-3)^2=-1((m-3)(m-3))=-1\left(m^2-3m-3m+9\right)=-1\left(m^2-6m+9\right)=\boxed{-m^2+6m-9}$$

c) $-2(3x-5)-(2x-1)^2$

Solution: We start with exponentiation

$$-2(3x-5)-(2x-1)^2=$$

$$(2x-1)^2=(2x-1)(2x-1)=4x^2-2x-2x+1=4x^2-4x+1$$

$$\begin{aligned}
&=-2(3x-5)-\left(4x^2-4x+1\right) && \text{distribute } -2\text{, to subtract is to add the opposite}\\
&=-6x+10+\left(-4x^2+4x-1\right) && \text{drop parentheses, combine like terms}\\
&=\boxed{-4x^2-2x+9}
\end{aligned}$$

4. $(x-3)^2-(2x-5)(x+1)=5-(x-1)^2$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{aligned}
(x-3)^2-(2x-5)(x+1) &= 5-(x-1)^2\\
x^2-3x-3x+9-\left(2x^2+2x-5x-5\right) &= 5-\left(x^2-x-x+1\right) && \text{combine like terms}\\
x^2-6x+9-\left(2x^2-3x-5\right) &= 5-\left(x^2-2x+1\right) && \text{distribute}\\
x^2-6x+9-2x^2+3x+5 &= 5-x^2+2x-1 && \text{combine like terms}\\
-x^2-3x+14 &= -x^2+2x+4 && \text{add } x^2\\
-3x+14 &= 2x+4 && \text{add } 3x\\
14 &= 5x+4 && \text{subtract } 4\\
10 &= 5x && \text{divide by } 5\\
2 &= x
\end{aligned}$$

We check. If $x=2$, then

$$\begin{aligned}
\text{LHS} &= (2-3)^2-(2\cdot 2-5)(2+1)=(-1)^2-(4-5)(2+1)=(-1)^2-(-1)\cdot 3\\
&= 1-(-3)=4\\
\text{RHS} &= 5-(2-1)^2=5-1^2=5-1=4
\end{aligned}$$

Thus $\boxed{x=2}$ is indeed the solution.

5. $(x+1)^2 - (2x-1)^2 + (3x)^2 = 6x(x-2)$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{aligned}
(x+1)^2 - (2x-1)^2 + (3x)^2 &= 6x(x-2) \\
x^2 + x + x + 1 - \left(4x^2 - 2x - 2x + 1\right) + 9x^2 &= 6x^2 - 12x \\
x^2 + 2x + 1 - \left(4x^2 - 4x + 1\right) + 9x^2 &= 6x^2 - 12x \qquad \text{distribute} \\
x^2 + 2x + 1 - 4x^2 + 4x - 1 + 9x^2 &= 6x^2 - 12x \qquad \text{combine like terms} \\
6x^2 + 6x &= 6x^2 - 12x \qquad \text{subtract } 6x^2 \\
6x &= -12x \qquad \text{add } 12x \\
18x &= 0 \qquad \text{divide by } 18 \\
x &= 0
\end{aligned}$$

We check. If $x = 0$, then

$$\begin{aligned}
\text{LHS} &= (0+1)^2 - (2\cdot 0 - 1)^2 + (3\cdot 0)^2 = 1^2 - (-1)^2 + (0)^2 = 1 - 1 + 0 = 0 \\
\text{RHS} &= 6\cdot 0\cdot (0-2) = 6\cdot 0\cdot (-2) = 0
\end{aligned}$$

Thus $\boxed{x = 0}$ is indeed the solution.

6. $12 - (2p-1)(p+1) = -2(-p+5)^2$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{aligned}
12 - (2p-1)(p+1) &= -2(-p+5)^2 \\
12 - \left(2p^2 + 2p - p - 1\right) &= -2\left(p^2 - 5p - 5p + 25\right) \qquad \text{combine like terms} \\
12 - \left(2p^2 + p - 1\right) &= -2\left(p^2 - 10p + 25\right) \qquad \text{distribute} \\
12 - 2p^2 - p + 1 &= -2p^2 + 20p - 50 \qquad \text{combine like terms} \\
-2p^2 - p + 13 &= -2p^2 + 20p - 50 \qquad \text{add } 2p^2 \\
-p + 13 &= 20p - 50 \qquad \text{add } p \\
13 &= 21p - 50 \qquad \text{add } 50 \\
63 &= 21p \qquad \text{divide by } 21 \\
3 &= p
\end{aligned}$$

We check. If $p = 3$, then

$$\begin{aligned}
\text{LHS} &= 12 - (2\cdot 3 - 1)(3+1) = 12 - (6-1)(3+1) = 12 - 5\cdot 4 = 12 - 20 = -8 \\
\text{RHS} &= -2(-3+5)^2 = -2\cdot 2^2 = -2\cdot 4 = -8
\end{aligned}$$

Thus $\boxed{p = 3}$ is indeed the solution.

7. If we increase each side of a square by $3\,\mathrm{cm}$, its area increases by $51\,\mathrm{cm}^2$. How long are the sides before the increase?

   Solution: Let us denote the length of the sides of the original square by x. Then the area of the square is x^2. The side of the larger square is $x+3$. Therefore, the area of the larger square is $(x+3)^2$. The equation will express the comparison between the two areas.expand complete square on the left-hand side

$$\begin{aligned} (x+3)^2 &= x^2+51 && \text{expand complete square on the left-hand side} \\ x^2+6x+9 &= x^2+51 && \text{subtract } x^2 \\ 6x+9 &= 51 && \text{subtract } 9 \\ 6x &= 42 && \text{divide by } 6 \\ x &= 7 \end{aligned}$$

   Thus the original square has sides $7\,\mathrm{cm}$ long. The area is $49\,\mathrm{cm}^2$. If we increased each side by $3\,\mathrm{cm}$, the new side is $10\,\mathrm{cm}$, and the new area $100\,\mathrm{cm}^2$. Indeed, the two areas differ by $100\,\mathrm{cm}^2 - 49\,\mathrm{cm}^2 = 51\,\mathrm{cm}^2$. Thus our solution is correct: the original square has sides that are $\boxed{7\,\mathrm{cm}}$ long.

Discussion: Explain why re-writing $2(x-3)^2$ as $(2x-6)^2$ would be an incorrect step.

Solution: If we don't immediately see what is going on with abstract algebraic expressions, it might be useful to make things concrete by looking at the numbers this expressions become for a value of x. If $x=0$, then the value of $2(x-3)^2 = 2(-3)^2 = 2\cdot 9 = 18$ and the value of $(2x-6)^2 = (-6)^2 = 36$ so the two expressions can not be equivalent. We can check a few more values using any numbers for x, we see that the number in the third column is stubbornly twice the number in the second column.

x	$2(x-3)^2$	$(2x-6)^2$
0	18	36
5	8	16
-1	32	64
10	98	196

$(2x-6)^2$ can be re-written as follows.

$$(2x-6)^2 = (2x-6)(2x-6) = [2(x-3)][2(x-3)] = 4(x-3)(x-3) = 4(x-3)^2$$

So, $2(x-3)^2$ can not be simplified as $(2x-6)^2$. We would be multiplying by 4 instead of 2.

