This handout will provide a quick review of operations with algebraic expressions. For a more thorough review, please see an introductory algebra book.

Definition: A **numerical expression** is an expression that combines numbers and operations.

For example, $3 \cdot 5^2$ is a numerical expression. So are $-\dfrac{2}{3+1}$ and $\sqrt{16}$ and $-\left|-5\right|$. We can **evaluate** numerical expressions by correctly applying the order of operations agreement. It is important that we clearly understand notation.

Example 1: Evaluate each of the given numerical expressions.

a) $3 \cdot 5^2$ b) $-\dfrac{2}{3+1}$ c) 3^2+2^2 d) $(3+2)^2$ e) -3^2 f) $(-3)^2$ g) $-\left|-5\right|$

Solution: a) Between exponentiation and multiplication, we first perform the exponentiation. $3 \cdot 5^2 = 3 \cdot 25 = \boxed{75}$

b) The addition in the denominator must be performed before we divide. (Why?) $-\dfrac{2}{3+1} = -\dfrac{2}{4} = \boxed{-\dfrac{1}{2}}$

c) $3^2+2^2 = 9+4 = \boxed{13}$

d) $(3+2)^2 = 5^2 = \boxed{25}$

Note: The error of confusing 3^2+2^2 with $(3+2)^2$ is called the "Freshman's Dream Error".

e) $-3^2 = \boxed{-9}$

f) $(-3)^2 = \boxed{9}$

Note: In looking at -3^2 and $(-3)^2$, we can interpret the minus sign in front of 3 as 'the opposite of'. That is the same as multiplication by -1. Now we can apply order of opertaions, and exponentiation comes before multiplication.

$$-3^2 = -1 \cdot 3^2 = -1 \cdot 9 = -9 \quad \text{but} \quad (-3)^2 = (-3)(-3) = 9$$

In the case of -3^2, we take the opposite of the square of three.
In the case of $(-3)^2$, we square the opposite of three.

g) $-\left|-5\right| = \boxed{-5}$ This is a perfect example that two minuses don't always make a plus. What happens here?

Definition: An **algebraic expression** is an expression that combines numbers, operations, and variables.

Variables always represent numbers, so they are subjects to the same rules as numbers. For example, $3x^2-1$ is an algebraic expression. So are $-x+3$ and $2a-b$ and $5y+3$. We can not automatically evaluate an algebraic expression because we often do not know the value of the variables. For example, the expression $3x^2-1$ has different values for different values of x.

Example 2: Evaluate the algebraic expression $3x^2-1$ given the values of x.

a) $x=2$ b) $x=0$ c) $x=-1$ d) $x=\dfrac{1}{2}$

Solution: a) If $x=2$, then $3x^2-1 = 3 \cdot 2^2 - 1 = 3 \cdot 4 - 1 = 12 - 1 = \boxed{11}$

b) If $x=0$, then $3x^2-1 = 3 \cdot 0^2 - 1 = 3 \cdot 0 - 1 = 0 - 1 = \boxed{-1}$

c) If $x=-1$, then $3x^2-1 = 3(-1)^2 - 1 = 3 \cdot 1 - 1 = 3 - 1 = \boxed{2}$

d) If $x=\frac{1}{2}$, then $3x^2-1=3\left(\frac{1}{2}\right)^2-1=3\left(\frac{1}{2}\cdot\frac{1}{2}\right)-1=3\cdot\frac{1}{4}-1=\frac{3}{4}-1=\frac{3}{4}-\frac{4}{4}=\boxed{-\frac{1}{4}}$

When we don't know the value of a variable, we often need to **simplify** algebraic expressions. Consider, for example, the algebraic expression $2x+3x$. Clearly, we can simplify $2x+3x$ and just write $5x$ instead. This is also called **combining like terms.**

Example 3: Simplify each of the following by combining like terms.

a) $3x-2-10x+x+7$ b) $3x^2-2x-4-8x^2+x+1$ c) $ab-a^2+b^2$ d) $\frac{1}{2}x^2-3+\frac{1}{2}x^2-4-x^2$

Solution: a) $3x$, $-10x$, and x are like terms, and -2 and 7 are like terms.
To combine like terms, we add the numbers (sign included!) that are multiplying the variable(s). Such a number is called the **coefficient**. When combining like terms, we add the coefficients.

To combine $3x$, $-10x$, and x, we add the coefficients: $3x-10x+x=(3-10+1)\,x=-6x$
Caution! It is a common mistake to misinterpret $3-10+1$ as $3-11$. Not so!

To combine -2 and 7, we just add and so we get 5. The entire process can be done mentally, so our computation will look like this:

$$3x-2-10x+x+7=\boxed{-6x+5}$$

b) $3x^2-2x-4-8x^2+x+1=3x^2-8x^2-2x+x-4+1=\boxed{-5x^2-x-3}$
-3, $-x$, and $-5x^2$ are unlike terms and so this expression can not be further simplified.

c) $\boxed{ab-a^2+b^2}$ since all three terms are unlike, so this expression can not be simplified.

d) $\frac{1}{2}x^2-3+\frac{1}{2}x^2-4-x^2=\left(\frac{1}{2}+\frac{1}{2}\right)x^2-3-4-x^2=1x^2-7-x^2=\boxed{-7}$

In the last example, as we simplified the expression, the variable disappeared. This special and often celebrated case of combining like terms is what we call **cancellation**.

To add two or more algebraic expressions, we drop parentheses and combine like terms.

Example 4: Add the algebraic expressions as indicated.

a) $(3a-5b)+(2a-b)$ b) $\left(-x^2+3x-4\right)+\left(5x^2-x-4\right)$ c) $(3y+5)+(3y-5)$ d) $\left(\frac{1}{2}x-\frac{2}{5}\right)+\left(\frac{3}{2}x+\frac{8}{5}\right)$

Solution: a) $(3a-5b)+(2a-b)=3a-5b+2a-b=3a+2a-5b-b=\boxed{5a-6b}$

b) $\left(-x^2+3x-4\right)+\left(5x^2-x-4\right)=-x^2+5x^2+3x-x-4-4=\boxed{4x^2+2x-8}$

c) $(3y+5)+(3y-5)=3y+5+3y-5=3y+3y+5-5=\boxed{6y}$

d) $\left(\frac{1}{2}x-\frac{2}{5}\right)+\left(\frac{3}{2}x+\frac{8}{5}\right)=\frac{1}{2}x-\frac{2}{5}+\frac{3}{2}x+\frac{8}{5}=\frac{1}{2}x+\frac{3}{2}x+\frac{8}{5}-\frac{2}{5}=\left(\frac{1}{2}+\frac{3}{2}\right)x+\frac{8}{5}-\frac{2}{5}=\frac{4}{2}x+\frac{6}{5}=\boxed{2x+\frac{6}{5}}$

To multiply an algebraic expression by a number or a one-term expression, we apply the distributive law.

Example 5: Expand the products as indicated.

a) $3\left(5x^2 - x + 1\right)$ b) $-1\left(-x^2 + 3x - 4\right)$ c) $5x\left(2a - x\right)$ d) $-ab\left(3a - 5b - 1\right)$ e) $\frac{2}{5}\left(\frac{3}{4}m - \frac{1}{2}\right)$

Solution: a) $3\left(5x^2 - x + 1\right) = \boxed{15x^2 - 3x + 3}$

b) $-1\left(-x^2 + 3x - 4\right) = \boxed{x^2 - 3x + 4}$

Note that multipliction by -1 means we change the sign in front of each term.

c) $5x\left(2a - x\right) = 5x \cdot 2a - 5x \cdot x = 10ax - 5x^2 = \boxed{-5x^2 + 10ax}$

d) $-ab\left(3a - 5b - 1\right) = -ab \cdot 3a - ab\left(-5b\right) - ab\left(-1\right) = \boxed{-3a^2b + 5ab^2 + ab}$

e) $\frac{2}{5}\left(\frac{3}{4}m - \frac{1}{2}\right) = \frac{2}{5} \cdot \frac{3}{4}m + \frac{2}{5}\left(-\frac{1}{2}\right) = \boxed{\frac{3}{10}m - \frac{1}{5}}$

To subtract an algebraic expression, we add the opposite.

Example 6: Perform the subtractions between algebraic expressions as indicated.

a) $(3a - 5b) - (2a - b)$ b) $\left(-x^2 + 3x - 4\right) - \left(5x^2 - x - 4\right)$ c) $(3y + 5) - (3y - 5)$ d) $\left(\frac{1}{2}x - \frac{2}{5}\right) - \left(\frac{3}{2}x + \frac{8}{5}\right)$

Solution: a) We apply one fundamental fact of algebra: *To subtract is to add the opposite.* The opposite is always obtained by multiplication by -1. And this multiplication by -1 means we need to distribute -1. We subtract the entire expression, not just its first term. Here is the argument, broken down to logical steps.

$$\begin{aligned}
(3a - 5b) - (2a - b) &= && \text{to subtract is to add the opposite,}\\
(3a - 5b) + (-1)(2a - b) &= && \text{the opposite is obtained by multiplying by } -1 \text{ (careful with the distributive law)}\\
(3a - 5b) + (-2a + b) &= && \text{we add the algebraic expressions by dropping the parentheses}\\
3a - 5b - 2a + b &= && \text{and combine like terms}\\
3a - 2a - 5b + b &= && \boxed{a - 4b}
\end{aligned}$$

But this is way too much writing. While the idea is the same, our computation usually looks like this:

$$(3a - 5b) - (2a - b) = (3a - 5b) + (-2a + b) = a - 4b$$

Careful! When computing on paper, we advise *not to subtract mentally*. To subtract is to add the opposite. Take the time of writing down the opposite, and then *add* mentally.

b) $\left(-x^2 + 3x - 4\right) - \left(5x^2 - x - 4\right) = \left(-x^2 + 3x - 4\right) + \left(-5x^2 + x + 4\right) = \boxed{-6x^2 + 4x}$

c) $(3y + 5) - (3y - 5) = (3y + 5) + (-3y + 5) = \boxed{10}$

d) $\left(\frac{1}{2}x - \frac{2}{5}\right) - \left(\frac{3}{2}x + \frac{8}{5}\right) = \left(\frac{1}{2}x - \frac{2}{5}\right) + \left(-\frac{3}{2}x - \frac{8}{5}\right) = \left(\frac{1}{2} - \frac{3}{2}\right)x - \frac{2}{5} - \frac{8}{5} = \frac{-2}{2}x - \frac{10}{5} = \boxed{-x - 2}$

We can now combine more complicated expressions.

Example 7: Simplify each of the following expressions.

a) $4(a - 2b + 1) - 5(2a - b - 1)$ b) $\left(-x^2 + x - 2\right)4 - 3\left(5x^2 - x + 1\right)$ c) $6(2y + 1) - 5(3y - 5)$

d) $12\left(\frac{1}{2}x - \frac{3}{4}\right) - 4\left(\frac{3}{2}x - \frac{5}{4}\right)$

Solution: a) We apply the distributive law and then combine like terms.

$$4(a-2b+1)-5(2a-b-1)=4a-8b+4-10a+5b+5=\boxed{-6a-3b+9}$$

b) Since $\left(-x^2+x-2\right)4$ looks rather odd, we immediately re-write it as $4\left(-x^2+x-2\right)$

$$\begin{aligned}\left(-x^2+x-2\right)4-3\left(5x^2-x+1\right) &= 4\left(-x^2+x-2\right)-3\left(5x^2-x+1\right)\\ &= -4x^2+4x-8-15x^2+3x-3\\ &= \boxed{-19x^2+7x-11}\end{aligned}$$

c) $6(2y+1)-5(3y-5)=12y+6-15y+25=\boxed{-3y+31}$

d) $12\left(\frac{1}{2}x-\frac{3}{4}\right)-4\left(\frac{3}{2}x-\frac{5}{4}\right)=12\cdot\frac{1}{2}x-12\cdot\frac{3}{4}-4\cdot\frac{3}{2}x-4\cdot\left(-\frac{5}{4}\right)=6x-9-6x+5=\boxed{-4}$

The last example illustrates the benefits of algebra. If we were asked to evaluate the expression $12\left(\frac{1}{2}x-\frac{3}{4}\right)-4\left(\frac{3}{2}x-\frac{5}{4}\right)$ when $x=8, -20, \frac{3}{5}$, or any other number, we might be computing for minutes, but the result will always be -4. In other words, this expression is **equivalent** to -4. It is a natural instinct to present and think of expressions in their simplest possible form.

When multiplying algebraic expressions, we apply the distributive law and then combine like terms.

Example 8: Expand each of the following.

a) $(x-3)(-2x+5)$ b) $(3a-2b)(5x-y)$ c) $(2y+1)(3y-5)$ d) $\left(\frac{1}{2}x-\frac{2}{3}\right)\left(4x-\frac{1}{2}\right)$

Solution: When we multiply two two-term expressions, FOIL is just one of the possible ways to make sure we applied the distributive law. (F - first with first; O - the outer terms; I - the inner terms; L - last with last)

a)
$$\begin{aligned}(x-3)(-2x+5) &= \overbrace{x\cdot(-2x)}^{F}+\overbrace{x\cdot 5}^{O}+\overbrace{(-3)\cdot(-2x)}^{I}+\overbrace{(-3)\cdot 5}^{L}\\ &= -2x^2+5x+6x-15 \qquad 5x \text{ and } 6x \text{ are like terms}\\ &= \boxed{-2x^2+11x-15}\end{aligned}$$

b)
$$\begin{aligned}(3a-2b)(5x-y) &= \overbrace{3a\cdot 5x}^{F}+\overbrace{3a\cdot(-y)}^{O}+\overbrace{(-2b)\cdot(5x)}^{I}+\overbrace{(-2b)\cdot(-y)}^{L}\\ &= 15ax-3ay-10bx+2by \qquad \text{all terms are unlike}\\ &= \boxed{15ax-3ay-10bx+2by}\end{aligned}$$

c)
$$\begin{aligned}(2y+1)(3y-5) &= \overbrace{2y\cdot 3y}^{F}+\overbrace{2y\cdot(-5)}^{O}+\overbrace{1\cdot 3y}^{I}+\overbrace{1\cdot(-5)}^{L}\\ &= 6y^2-10y+3y-5 \qquad -10y \text{ and } 3y \text{ are like terms}\\ &= \boxed{6y^2-7y-5}\end{aligned}$$

d) $$\left(\frac{1}{2}x-\frac{2}{3}\right)\left(4x-\frac{1}{2}\right) = \overbrace{\frac{1}{2}x\cdot 4x}^{\text{F}} + \overbrace{\frac{1}{2}x\left(-\frac{1}{2}\right)}^{\text{O}} + \overbrace{\left(-\frac{2}{3}\right)\cdot 4x}^{\text{I}} + \overbrace{\left(-\frac{2}{3}\right)\left(-\frac{1}{2}\right)}^{\text{L}}$$
$$= 2x^2 - \frac{1}{4}x - \frac{8}{3}x + \frac{1}{3} \qquad -\frac{1}{4}x \text{ and } -\frac{8}{3}x \text{ are like terms}$$
$$= 2x^2-\frac{1}{4}x-\frac{8}{3}x+\frac{1}{3} = 2x^2-\frac{3}{12}x-\frac{32}{12}x+\frac{1}{3} = \boxed{2x^2-\frac{35}{12}x+\frac{1}{3}}$$

If the multiplication is more complicated, the distributive law is applied so that each term from one expression is multiplied by each term of the other expression exactly once.

Example 9: Expand each of the following.

a) $(x+2)\left(3x^2-5x-7\right)$ b) $(2a-b-1)(5a-4b)$

Solution: a) We will first multiply all three terms of the second expression by x, and then by 2. We expect six little products. Finally, we combine like terms.

$$(x+2)\left(3x^2-5x-7\right) = x\cdot 3x^2+x\cdot(-5)x+x\cdot(-7)+2\cdot 3x^2+2\cdot(-5x)+2\cdot(-7)$$
$$= 3x^3-5x^2-7x+6x^2-10x-14 = \boxed{3x^3+x^2-17x-14}$$

b) $$(2a-b-1)(5a-4b) = 2a\cdot 5a+2a\cdot(-4b)+(-b)\,5a+(-b)(-4b)+(-1)\cdot 5a+(-1)(-4b)$$
$$= 10a^2-8ab-5ab+4b^2-5a+4b \qquad -8ab \text{ and } -5ab \text{ are like terms.}$$
$$= \boxed{10a^2-13ab+4b^2-5a+4}$$

Example 10: Expand each of the following.

a) $(x+1)^2$ b) $(3a-2)^2$ c) $(2y-1)^2$ d) $(2y-1)^3$ e) $(3x-1)(3x+1)$ f) $(a+b)(a-b)$

Solution: When we square an entire sum or difference, the expression is called a **complete square**.

a) $(x+1)^2=(x+1)(x+1)=x\cdot x+x\cdot 1+1\cdot x+1\cdot 1=x^2+x+x+1=\boxed{x^2+2x+1}$

b) $(3a-2)^2=(3a-2)(3a-2)=9a^2-6a-6a+4=\boxed{9a^2-12a+4}$

c) $(2y-1)^2=(2y-1)(2y-1)=4y^2-2y-2y+1=\boxed{4y^2-4y+1}$

d) In solving this problem, we will apply our result from the previous problem.

$$(2y-1)^3 = (2y-1)(2y-1)(2y-1)=(2y-1)(2y-1)^2=(2y-1)\left(4y^2-4y+1\right)$$
$$= 8y^3-8y^2+2y-4y^2+4y-1=\boxed{8y^3-12y^2+6y-1}$$

e) The expressions $3x-1$ and $3x+1$ are **conjugates** of each other. Conjugates have very useful properties.
$(3x-1)(3x+1)=9x^2+3x-3x-1=\boxed{9x^2-1}$

f) $(a+b)(a-b)=a^2-ab+ba-b^2=a^2-ab+ab-b^2=\boxed{a^2-b^2}$

Discussion: Explain why re-writing $2(x-3)^2$ as $(2x-6)^2$ would be an incorrect step.

Example 11: Simplify the expression $(2x-3)^2-(2x+1)(3x-7)$

Solution: If we consider the expression as one that connects algebraic expressions, then there are three operations: the squaring, a multiplication, and a subtraction. We apply order of operations. Exponentiation first, then multiplication, and finally subtraction.

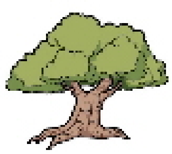
$(2x-3)^2-(2x+1)(3x-7)=$ exponentiation first: $(2x-3)^2=(2x-3)(2x-3)$
$=4x^2-6x-6x+9$
$=4x^2-12x+9$

$=\left(4x^2-12x+9\right)-(2x+1)(3x-7)$ multiplication next: $(2x+1)(3x-7)=6x^2-14x+3x-7$
$=6x^2-11x-7$

$=\left(4x^2-12x+9\right)-\left(6x^2-11x-7\right)=$ to subtract is to add the opposite

$=\left(4x^2-12x+9\right)+\left(-6x^2+11x+7\right)=\boxed{-2x^2-x+16}$

Caution! This example contains a situation, in which students are at risk to make an error. After we expand an expression such as $(2x+1)(3x-7)$, we usually do not need a parentheses. However, in this case, the product is subtracted and so we still need a pair of parentheses indicating that we are subtracting the entire expression and not just its first term. This is just one of the several examples in which we are tempted to violate the distributive law.



Enrichment

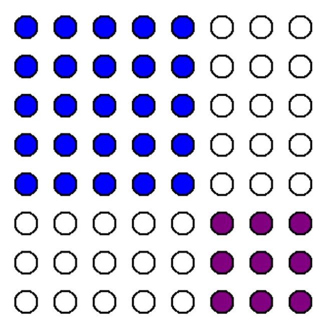
1. Special products. Expand each of the following.

   a) $(a+b)^2$ c) $(a+b)^3$ e) $(x-y)(x+y)$ g) $(x-y)\left(x^3+x^2y+xy^2+y^3\right)$

   b) $(a-b)^2$ d) $(a-b)^3$ f) $(x-y)\left(x^2+xy+y^2\right)$

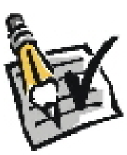
2. Explain how $2x+3x=5x$ can be explained in terms of the distributive law.

3. The *Freshman's Dream Error*. We saw that $3^2+5^2=9+25=34$ and $(3+5)^2=8^2=64$. To confuse a^2+b^2 - a sum of two squares with the complete square $(a+b)^2$ is an error so egregious, it actually has a name: The Freshman's Dream. The Freshman's Dream is a violation of the distributive law in which we state that $(a+b)^2=a^2+b^2$. Use the picture given to illustrate the error and the correct formula for the expansion of $(a+b)^2$.



Sample Problems

1. Evaluate each of the following numerical expressions.
   a) $2-5(3-7)$ c) -4^2 e) $|3|-|8|$ g) $\sqrt{25}-\sqrt{16}$
   b) $24-10+2$ d) $(-4)^2$ f) $|3-8|$ h) $\sqrt{25-16}$
2. Evaluate each of the following algebrac expressions with the value(s) given.
   a) $-x^2-5x+2$ if $x=-2$ b) $-16t^2+32t+240$ if $t=3$
3. Add the algebraic expressions as indicated.
   a) $(3x-5y)+(2x+4y)$ b) $\left(2x^2-5x+3\right)+\left(-x^2-8x+3\right)$
4. Multiply the algebraic expressions by a number as indicated.
   a) $3\left(2x^2+4y-5\right)$ b) $-5\left(2x^2-x+8\right)$ c) $-1\left(-x+3y-8z^2+6\right)$ d) $-\left(-a+3b^2-2ab+7\right)$
5. Subtract the algebraic expressions as indicated.
   a) $(2a+b)-(a-b)$ c) $\left(2x^2-5x+3\right)-\left(-x^2-8x+3\right)$ e) $(3a-2)-(1-4a)$
   b) $(3x-5y)-(2x+4y)$ d) $(2a-2b)-(b-a)$
6. Simplify each of the following.
   a) $3(x-5)-5(x-1)$ c) $2(a-2b)+3(5b-2a)-4(2b-a)$ e) $-(3a-2)-(1-4a)$
   b) $4(2a-b)-3(5a-2b)$ d) $\frac{5}{2}(x-y)+\frac{1}{2}(x-3y)$
7. Multiply the algebraic expressions as indicated.
   a) $(x+3)(5x-3)$ b) $(5-2x)^2$ c) $(x+4)(1-2x)$ d) $-2(x-3)^2$
8. Simplify each of the following expressions.
   a) $(x-5)^2-(2x-1)(x+3)$ b) $-(m-3)^2$ c) $-2(3x-5)-(2x-1)^2$



Practice Problems

1. Evaluate each of the following numerical expressions.
   a) $24-5+1$ c) -1^2 e) $-|4|-|7|$ g) 6^2-4^2 i) $\sqrt{25+144}$
   b) $24\div 3\cdot 2$ d) $(-1)^2$ f) $-|4-7|$ h) $(6-4)^2$ j) $\sqrt{25}+\sqrt{144}$
2. Evaluate each of the following algebrac expressions with the value(s) given.
   a) $3x^2-x+7$ if $x=-1$ b) $-a+5b$ if $a=3$ and $b=-2$ c) $\dfrac{x^x-1}{x-1}$ if $x=2$
3. Add the algebraic expressions as indicated.
   a) $\left(2a-7y^2\right)+\left(2a-7y^2\right)$ d) $(4p-5q)+(5p-4q)$ g) $(-5y+8)+\left(5y^2-8\right)$
   b) $(-2a-5)+(3a+5)$ e) $\left(2t^2-3t+8\right)+\left(-2t^2-3t-8\right)$
   c) $\left(2x^2-5x-1\right)+(-x+1)$ f) $\left(4a^2+9\right)+\left(4a^2-9\right)$
4. Multiply the algebraic expressions by a number as indicated.
   a) $5\left(-3ab^2+y-5\right)$ b) $0\left(2x^2-7x+8\right)$ c) $-16\left(t^2-6t-16\right)$ d) $-(x-5a+3ax-1)$

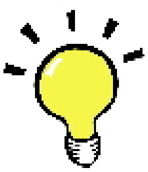
5. Subtract the algebraic expressions as indicated.
   a) $\left(2a-7y^2\right)-\left(2a-7y^2\right)$ d) $(4p-5q)-(5p-4q)$ g) $(-5y+8)-\left(5y^2-8\right)$
   b) $(-2a-5)-(3a+5)$ e) $\left(2t^2-3t+8\right)-\left(-2t^2-3t-8\right)$
   c) $\left(2x^2-5x-1\right)-(-x+1)$ f) $\left(4a^2+9\right)-\left(4a^2-9\right)$

6. Simplify each of the following.
   a) $(x+1)-(x-1)$ c) $-2+x-3(x-1)-(1-2x)$ e) $\dfrac{2}{3}(n-4m)-\dfrac{1}{3}(5n+m)$
   b) $2(x-1)-3(x-7)$ d) $-2\left(x^2-3x+1\right)-3\left(x^2-x+4\right)$

7. Multiply the algebraic expressions as indicated.
   a) $(3-2x)(x-7)$ b) $(3-2x)(3x-2)$ c) $(3-2x)(3+2x)$ d) $(3-2x)^2$ e) $2(a-1)^2$

8. Simplify each of the following. Notice what is the same and what is different in the problems.
   a) $(-2x+5)+(3x-8)$ b) $(-2x+5)-(3x-8)$ c) $3(-2x+5)-2(3x-8)$ d) $(-2x+5)(3x-8)$

9. Simplify each of the following expressions.
   a) $(x+2)^2-(x-2)^2$ b) $1-(x-4)(3x-1)$ c) $-3(5m-1)^2$ d) $3x-1-3x(3x-1)$



Answers

Sample Problems

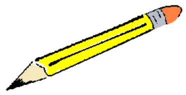
1. a) 22 b) 16 c) -16 d) 16 e) -5 f) 5 g) 1 h) 3 2. a) 8 b) 192 3. a) $5x-y$ b) $x^2-13x+6$
4. a) $6x^2+12y-15$ b) $-10x^2+5x-40$ c) $x-3y+8z^2-6$ d) $a-3b^2+2ab-7$
5. a) $a+2b$ b) $x-9y$ c) $3x^2+3x$ d) $3a-3b$ e) $7a-3$
6. a) $-2x-10$ b) $-7a+2b$ c) $3b$ d) $3x-4y$ e) $a+1$
7. a) $5x^2+12x-9$ b) $4x^2-20x+25$ c) $-2x^2-7x+4$ d) $-2x^2+12x-18$
8. a) $-x^2-15x+28$ b) $-m^2+6m-9$ c) $-4x^2-2x+9$

Practice Problems

1. a) 20 b) 16 c) -1 d) 1 e) -11 f) -3 g) 20 h) 4 i) 13 j) 17 2. a) 11 b) -13 c) 3
3. a) $4a-14y^2$ b) a c) $2x^2-6x$ d) $9p-9q$ e) $-6t$ f) $8a^2$ g) $5y^2-5y$
4. a) $-15ab^2+5y-25$ b) 0 c) $-16t^2+96t+256$ d) $-x+5a-3ax+1$
5. a) 0 b) $-5a-10$ c) $2x^2-4x-2$ d) $-p-q$ e) $4t^2+16$ f) 18 g) $-5y^2-5y+16$
6. a) 2 b) $-x+19$ c) 0 d) $-5x^2+9x-14$ e) $-3m-n$
7. a) $-2x^2+17x-21$ b) $-6x^2+13x-6$ c) $-4x^2+9$ d) $4x^2-12x+9$ e) $2a^2-4a+2$
8. a) $x-3$ b) $-5x+13$ c) $-12x+31$ d) $-6x^2+31x-40$
9. a) $8x$ b) $-3x^2+13x-3$ c) $-75m^2+30m-3$ d) $-9x^2+6x-1$

Answers - Enrichment

1. a) $(a+b)^2=a^2+2ab+b^2$ b) $(a-b)^2=a^2-2ab+b^2$ c) $(a+b)^3=a^3+3a^2b+3ab^2+b^3$
   d) $(a-b)^3=a^3-3a^2b+3ab^2-b^3$ e) $(x-y)(x+y)=x^2-y^2$
   f) $(x-y)\left(x^2+xy+y^2\right)=x^3-y^3$ g) $(x-y)\left(x^3+x^2y+xy^2+y^3\right)=x^4-y^4$
2. $5x=x\cdot 5=x(2+3)=2x+3x$
3. Let $a=5$ and $b=3$. The expression a^2+b^2 or 5^2+3^2 is the combined region of the blue square and the purple square.
   On the other hand, $(5+3)^2$ can be seen as the entire region, the 8 by 8 square. Clearly $34\neq 64$.
   $(a+b)^2$ is the entire image (Still $a=5$ and $b=3$)
   a^2 is the blue region, b^2 is the purple region. The white rectangular dots represent ab and ba
   So, $(a+b)^2=a^2+b^2+2ab$



Solutions - Sample Problems

1. Evaluate each of the following numerical expressions.

a) $2-5(3-7)$

Solution: We will apply order of operations. First we perform the subtraction in the parentheses.

$$\begin{aligned} 2-5(3-7) &= \quad \text{subtraction in parentheses} \\ 2-5(-4) &= \quad \text{multiplication} \\ 2-(-20) &= \quad \text{subtraction} \\ 2+20 &= \boxed{22} \end{aligned}$$

b) $24-10+2$

Solution: It is NOT true that addition comes before subtraction. Addition and subtraction are equally strong, so between those two, we perform them left to right. First come, first served.

$$24-10+2=14+2=\boxed{16}$$

c) -4^2

Solution: as it was discussed before, -4^2 is quite different from $(-4)^2$. This is $-1\cdot 4^2=\boxed{-16}$.

d) $(-4)^2$

This is when -4 is squared. So $(-4)^2=-4(-4)=\boxed{16}$

e) $|3|-|8|$

Solution: We subtract the absolute value of 8 from the absolute value of 3. So $|3|-|8|=3-8=\boxed{-5}$

f) $|3-8|$

Solution: This is the absolute value of the difference. Absolute value signs also function of grouping symbols (i.e. parentheses) to overwrite the usual order of operations. So $|3-8|=|-5|=\boxed{5}$

g) $\sqrt{25}-\sqrt{16}$

Solution: $\sqrt{25}-\sqrt{16}=5-4=\boxed{1}$

h) $\sqrt{25-16}$

Solution: A radical stretching over an entire expression also serves as a parentheses. So, we subtract first and then take the square root. $\sqrt{25-16}=\sqrt{9}=\boxed{3}$

2. Evaluate each of the following algebrac expressions with the value(s) given.

a) $-x^2-5x+2$ if $x=-2$

Solution: We substitute -2 into the expression. Please note that if the value of x is negative, we will need to place parentheses around it.

$$-x^2-5x+2=-(-2)^2-5(-2)+2$$

According to order of operations, we perform the exponentiation first.

$$-(-2)^2-5(-2)+2=-4-5(-2)+2=-4-(-10)+2=-4+10+2=6+2=\boxed{8}$$

Why don't the two minuses make a plus in $-(-2)^2$? They do, it's just that there are three minus signs and not two: $-(-2)^2=-1(-2)(-2)$.

b) $-16t^2 + 32t + 240$ if $t = 3$

Solution: We substitute 3 into the expression. Please note that if the value of x is negative, we will need to place parentheses around it.

$$\begin{aligned} -16t^2 + 32t + 240 &= -16\,(3)^2 + 32\,(3) + 240 \\ &= -16 \cdot 9 + 32 \cdot 3 + 240 \\ &= -144 + 96 + 240 = -48 + 240 = \boxed{192} \end{aligned}$$

3. Add the algebraic expressions as indicated.

a) $(3x - 5y) + (2x + 4y)$

Solution: We drop the parentheses and combine like terms.

$$\begin{aligned} (3x - 5y) + (2x + 4y) &= \\ 3x - 5y + 2x + 4y &= && \text{organize like terms together} \\ 3x + 2x - 5y + 4y &= && 3 + 2 = 5 \quad \text{and} \quad -5 + 4 = -1 \\ &= \boxed{5x - y} \end{aligned}$$

b) $\left(2x^2 - 5x + 3\right) + \left(-x^2 - 8x + 3\right)$

Solution: We drop the parentheses and combine like terms.

$$\begin{aligned} \left(2x^2 - 5x + 3\right) + \left(-x^2 - 8x + 3\right) &= \\ 2x^2 - 5x + 3 - x^2 - 8x + 3 &= \\ 2x^2 - x^2 - 5x - 8x + 3 + 3 &= && 2 - 1 = 1, \quad -5 - 8 = -13, \quad \text{and} \quad 3 + 3 = 6 \\ 1x^2 - 13x + 6 &= \boxed{x^2 - 13x + 6} \end{aligned}$$

4. Multiply the algebraic expressions by a number as indicated.

a) $3\left(2x^2 + 4y - 5\right)$

Solution: We distribute 3.

$$3\left(2x^2 + 4y - 5\right) = \boxed{6x^2 + 12y - 15}$$

b) $-5\left(2x^2 - x + 8\right)$

Solution: We distribute -5.

$$-5\left(2x^2 - x + 8\right) = \boxed{-10x^2 + 5x - 40}$$

c) $-1\left(-x + 3y - 8z^2 + 6\right)$

Solution: We distribute -1.

$$-1\left(-x + 3y - 8z^2 + 6\right) = \boxed{x - 3y + 8z^2 - 6}$$

d) $-\left(-a + 3b^2 - 2ab + 7\right)$

Solution: The notation here indicates multiplication by -1, which is the same as taking the opposite of a quantity. We distribute -1.

$$-1\left(-a + 3b^2 - 2ab + 7\right) = \boxed{a - 3b^2 + 2ab - 7}$$

5. Subtract the algebraic expressions as indicated.

a) $(2a + b) - (a - b)$

Solution: To subtract is to add the opposite. The opposite of $a - b$ is $-a + b$ since

$$-1\,(a - b) = -a + b$$

$$\begin{aligned} \text{Thus} \quad (2a+b)-(a-b) &= (2a+b)+(-a+b) && \text{drop parentheses} \\ &= 2a+b-a+b && \text{combine like terms} \\ &= \boxed{a+2b} \end{aligned}$$

b) $(3x-5y)-(2x+4y)$

Solution: To subtract is to add the opposite. The opposite of $2x+4y$ is $-2x-4y$ since

$$-1\,(2x+4y) = -2x-4y$$

$$\begin{aligned} (3x-5y)-(2x+4y) &= (3x-5y)+(-2x-4y) && \text{to subtract is to add the opposite} \\ &= 3x-5y-2x-4y && \text{drop parentheses, combine like terms} \\ &= \boxed{x-9y} \end{aligned}$$

c) $\left(2x^2-5x+3\right)-\left(-x^2-8x+3\right)$

Solution: To subtract is to add the opposite. The opposite of $-x^2-8x+3$ is x^2+8x-3 since

$$-1\left(-x^2-8x+3\right) = x^2+8x-3$$

$$\begin{aligned} \left(2x^2-5x+3\right)-\left(-x^2-8x+3\right) &= && \text{to subtract is to add the opposite} \\ \left(2x^2-5x+3\right)+\left(x^2+8x-3\right) &= && \text{drop parentheses, combine like terms} \\ 2x^2-5x+3+x^2+8x-3 &= \boxed{3x^2+3x} \end{aligned}$$

d) $(2a-2b)-(b-a)$

Solution: To subtract is to add the opposite. The opposite of $b-a$ is $-b+a$ since

$$-1\,(b-a) = -b+a$$

$$\begin{aligned} (2a-2b)-(b-a) &= (2a-2b)+(-b+a) && \text{to subtract is to add the opposite} \\ &= 2a-2b-b+a && \text{drop parentheses, combine like terms} \\ &= \boxed{3a-3b} \end{aligned}$$

e) $(3a-2)-(1-4a)$

Solution: To subtract is to add the opposite. The opposite of $1-4a$ is $-1+4a$ since

$$-1\,(1-4a) = -1+4a$$

$$\begin{aligned} (3a-2)-(1-4a) &= && \text{to subtract is to add the opposite} \\ (3a-2)+(-1+4a) &= && \text{drop parentheses, combine like terms} \\ 3a-2-1+4a &= \boxed{7a-3} \end{aligned}$$

6. Simplify each of the following.

a) $3\,(x-5)-5\,(x-1)$

Solution: We apply the law of distributivity and combine like terms. Notice that the last term is $-5\,(-1) = 5$.

$$\begin{aligned} 3\,(x-5)-5\,(x-1) &= && \text{apply the distributie law} \\ 3x-15-5x+5 &= && \text{combine like terms} \\ &= \boxed{-2x-10} \end{aligned}$$

b) $4(2a-b)-3(5a-2b)$

Solution: We apply the law of distributivity and combine like terms

$$\begin{aligned} 4(2a-b)-3(5a-2b) &= 8a-4b-15a+6b \\ &= \boxed{-7a+2b} \end{aligned}$$

c) $2(a-2b)+3(5b-2a)-4(2b-a)$

Solution: We apply the law of distributivity and combine like terms

$$2(a-2b)+3(5b-2a)-4(2b-a)=2a-4b+15b-6a-8b+4a=\boxed{3b}$$

d) $\dfrac{5}{2}(x-y)+\dfrac{1}{2}(x-3y)$

Solution: We apply the law of distributivity and combine like terms

$$\begin{aligned} \frac{5}{2}(x-y)+\frac{1}{2}(x-3y) &= && \text{apply the distributive law} \\ \frac{5}{2}x-\frac{5}{2}y+\frac{1}{2}x-\frac{1}{2}\cdot 3y &= && \\ \frac{5}{2}x+\frac{1}{2}x-\frac{5}{2}y-\frac{3}{2}y &= && \text{combine like terms } \frac{5}{2}+\frac{1}{2}=3 \quad -\frac{5}{2}-\frac{3}{2}=-4 \\ \frac{6}{2}x-\frac{8}{2}y &= \boxed{3x-4y} && \end{aligned}$$

e) $-(3a-2)-(1-4a)$

Solution:

$$\begin{aligned} -(3a-2)-(1-4a) &= -1(3a-2)-1(1-4a) && \text{multiplication} \\ &= -3a+2-1+4a && \text{combine like terms} \\ &= \boxed{a+1} \end{aligned}$$

7. Multiply the algebraic expressions as indicated.

a) $(x+3)(5x-3)$

Solution: We expand the expression using the distributive law. In the simplest case, when both expressions have only two terms, we use FOIL (F - first term with first term, O - outer terms, I - inner terms, L - last terms)

$$\begin{aligned} (x+3)(5x-3) &= && \text{FOIL} \\ 5x^2-3x+15x-9 &= && \text{combine like terms} \\ &= \boxed{5x^2+12x-9} \end{aligned}$$

b) $(5-2x)^2$

Solution: To square something means to write it down twice and multiply.

$$\begin{aligned} (5-2x)^2 &= (5-2x)(5-2x) && \text{FOIL} \\ &= 25-10x-10x+4x^2 && \text{combine like terms} \\ &= \boxed{4x^2-20x+25} \end{aligned}$$

c) $(x+4)(1-2x)$

Solution: We expand the expression and combine like terms.

$$(x+4)(1-2x)=x-2x^2+4-8x=\boxed{-2x^2-7x+4}$$

d) $-2(x-3)^2$

Solution: We have two operations: multiplication by -2 and exponentiation. Order of operations still apply, thus we start with exponentiation.

$$-2(x-3)^2 = -2((x-3)(x-3)) = -2\left(x^2-3x-3x+9\right) = -2\left(x^2-6x+9\right) = \boxed{-2x^2+12x-18}$$

8. Simplify each of the following expressions.

a) $(x-5)^2-(2x-1)(x+3)$

Solution: If we consider this as operations on algebraic expressions, then we are faced with an exponentiation, a multiplication, and a subtraction. We will execute them exactly in this order.

$$(x-5)^2-(2x-1)(x+3) =$$

$$(x-5)^2 = (x-5)(x-5) = x^2-5x-5x+25 = x^2-10x+25$$

$$= x^2-10x+25-(2x-1)(x+3)$$

$$(2x-1)(x+3) = 2x^2+6x-x-3 = 2x^2+5x-3$$

$$= x^2-10x+25-\left(2x^2+5x-3\right)$$

$$= x^2-10x+25+\left(-2x^2-5x+3\right) = \boxed{-x^2-15x+28}$$

b) $-(m-3)^2$

Solution: We are asked to take the opposite of a complete square.

$$-(m-3)^2 = -1((m-3)(m-3)) = -1\left(m^2-3m-3m+9\right) = -1\left(m^2-6m+9\right) = \boxed{-m^2+6m-9}$$

c) $-2(3x-5)-(2x-1)^2$

Solution: We start with exponentiation

$$-2(3x-5)-(2x-1)^2 =$$

$$(2x-1)^2 = (2x-1)(2x-1) = 4x^2-2x-2x+1 = 4x^2-4x+1$$

$$= -2(3x-5)-\left(4x^2-4x+1\right)$$ distribute -2, to subtract is to add the opposite

$$= -6x+10+\left(-4x^2+4x-1\right)$$ drop parentheses, combine like terms

$$= \boxed{-4x^2-2x+9}$$

Discussion: Explain why re-writing $2(x-3)^2$ as $(2x-6)^2$ would be an incorrect step.

Solution: If we don't immediately see what is going on with abstract algebraic expressions, it might be useful to make things concrete by looking at the numbers this expressions become for a value of x. If $x=0$, then the value of $2(x-3)^2 = 2(-3)^2 = 2\cdot 9 = 18$ and the value of $(2x-6)^2 = (-6)^2 = 36$ so the two expressions can not be equivalent. We can check a few more values using any numbers for x, we see that the number in the third column is stubbornly twice the number in the second column.

x	$2(x-3)^2$	$(2x-6)^2$
0	18	36
5	8	16
-1	32	64
10	98	196

$(2x-6)^2$ can be re-written as follows.

$$(2x-6)^2 = (2x-6)(2x-6) = [2(x-3)][2(x-3)] = 4(x-3)(x-3) = 4(x-3)^2$$

So, $2(x-3)^2$ can not be simplified as $(2x-6)^2$. We would be multiplying by 4 instead of 2.

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