We will now continue studying algebraic expressions. In this section, we will learn how to simplify complicated algebraic expressions. We already know how to evaluate algebraic expressions. However, that is not always desirable or even possible. One frequently occuring example for this is when we cannot evaluate an algebraic expression becuse we do not know the value of the unknown. In such cases, we often need to **simplify** algebraic expressions. Consider, for example, the algebraic expression $2x+3x$. We can simplify $2x+3x$ and just write $5x$ instead. This is also called **combining like terms.**

Example 1. Simplify each of the following by combining like terms.

a) $3x-2-10x+x+7$ b) $3m-2n-4-8m+n+1$ c) $ab-a+b$ d) $p+q+p-q$

Solution: a) $3x$, $-10x$, and x are like terms, and -2 and 7 are like terms.

To combine like terms, we add the numbers (sign included!) that are multiplying the variable(s). Such a number is called the **coefficient**. When combining like terms, we add the coefficients.
To combine $3x$, $-10x$, and x, we add the coefficients: $3x-10x+x=(3-10+1)\,x=-6x$

Caution! It is a common mistake to misinterpret $3-10+1$ as $3-11$. Not so!

To combine -2 and 7, we just add and so we get 5. The entire process can be done mentally, so our computation will look like this:

$$3x-2-10x+x+7=\boxed{-6x+5}$$

b) $3m-2n-4-8m+n+1=3m-8m-2n+n-4+1=\boxed{-5m-n-3}$
-3, $-n$, and $-5m$ are unlike terms and so this expression can not be further simplified.

c) $\boxed{ab-a+b}$ since all three terms are unlike, so this expression can not be simplified.

d) $p+q+p-q=p+p+q-q=\boxed{2p}$

In the last example, as we simplified the expression, one of the variables disappeared. This special and often celebrated case of combining like terms is what we call **cancellation**.

> To add two or more algebraic expressions, we drop parentheses and combine like terms.

Example 2. Add the algebraic expressions as indicated.

a) $(3a-5b)+(2a-b)$ b) $(-m+3n-4)+(5m-n-4)$ c) $(3y+5)+(3y-5)$

Solution: a) $(3a-5b)+(2a-b)=3a-5b+2a-b=3a+2a-5b-b=\boxed{5a-6b}$

b) $(-m+3n-4)+(5m-n-4)=-m+5m+3n-n-4-4=\boxed{4m+2n-8}$

c) $(3y+5)+(3y-5)=3y+5+3y-5=3y+3y+5-5=\boxed{6y}$

> To multiply an algebraic expression by a number or a one-term expression, we apply the distributive law:
>
> $$a\,(b+c)=ab+ac \quad \text{for all numbers } a,\ b, \text{ and } c.$$

Example 3. Expand the products as indicated.

a) $3(5a-b+1)$ b) $-1\left(-x^2+3x-4\right)$ c) $5x(2a-x)$ d) $-ab(3a-5b-1)$ e) $-4(3m-5)$

Solution: a) $3(5a-b+1)=\boxed{15a-3b+3}$

b) $-1(-x+3y-4)=\boxed{x-3y+4}$

Note that multipliction by -1 means we change the sign in front of each term.

c) $5x(2a-3)=5x\cdot 2a-5x\cdot 3=10ax-15x=\boxed{10ax-15x}$

d) $-ab(3a-5b-1)=-ab\cdot 3a-ab(-5b)-ab(-1)=\boxed{-3a^2b+5ab^2+ab}$

e) $-4(3m-5)=\boxed{-12m+20}$

> To subtract an algebraic expression, we add the opposite.

Example 4. Perform the subtractions between algebraic expressions as indicated.

a) $(3a-5b)-(2a-b)$ b) $(3y+5)-(3y-5)$

Solution: a) We apply one fundamental fact of algebra: *To subtract is to add the opposite.* The opposite is always obtained by multiplication by -1. And this multiplication by -1 means we need to distribute -1. We subtract the entire expression, not just its first term. Here is the argument, broken down to logical steps.

$(3a-5b)-(2a-b)=$ to subtract is to add the opposite

$=(3a-5b)+(-1)(2a-b)$ the opposite is obtained by multiplying by -1 (careful with the distributive law)

$=(3a-5b)+(-2a+b)$ we add the algebraic expressions by dropping the parentheses

$=3a-5b-2a+b$ and combine like terms

$=3a-2a-5b+b=\boxed{a-4b}$

But this is way too much writing. While the idea is the same, our computation usually looks like this:

$$(3a-5b)-(2a-b)=3a-5b-2a+b=a-4b$$

Careful! When computing on paper, we advise *not to subtract mentally*. To subtract is to add the opposite. Take the time of writing down the opposite, and then *add* mentally.

b) $(3y+5)-(3y-5)=(3y+5)+(-3y+5)=\boxed{10}$

We can now combine more complicated expressions.

Example 5. Simplify each of the following expressions.

a) $4(a-2b+1)-5(2a-b-1)$ b) $6(2y+1)-5(3y-5)$ c) $3(5x-2)-5(3x+1)$

Solution: a) We apply the distributive law and then combine like terms.

$4(a-2b+1)-5(2a-b-1)=4a-8b+4-10a+5b+5=\boxed{-6a-3b+9}$

b) $6(2y+1)-5(3y-5)=12y+6-15y+25=\boxed{-3y+31}$

c) $3(5x-2)-5(3x+1)=15x-6-15x-5=\boxed{-11}$

The last example illustrates the benefits of algebra. If we were asked to evaluate the expression $3(5x-2)-5(3x+1)$ when $x=8$,-20, or -97, or any other number, we might be computing for minutes, but the result will always be -11. In other words, this expression is **equivalent** to -11. It is a natural instinct and *widely accepted convention to always present expressions in their simplest possible form*.

Discussion: Explain how $2x + 3x = 5x$ can be explained in terms of the distributive law.

Note: The equation $3a + 2a = 5a$ is an **identity**. An identity is an equation for which all numbers are solutions. We only check two values for a:

$$\begin{aligned} \text{If } a &= 5, \text{ then } 3 \cdot 5 + 2 \cdot 5 = 15 + 10 = 25 = 5 \cdot 5 \\ \text{If } a &= -2, \text{ then } 3(-2) + 2(-2) = -6 + (-4) = -10 = 5(-2) \end{aligned}$$

When we simplify algebraic expressions, equations such as $5x - 2x = 3x$ are always identities.

Example 6. The longer side of a rectangle is two feet shorter than three times the shorter side.

a) Suppose we label the shorter side by x. Express the other side in terms of x.

b) Express the perimeter of the rectangle in terms of x.

c) Simplify the expression expressing the perimeter.

Solution: a) If the shorter side is x, then the longer side is $\boxed{3x - 2}$.

b) For the perimeter, we need to add all four sides. Since the opposite sides are equal, the perimeter is

$$P = 2a + 2b = \boxed{2x + 2(3x - 2)}.$$

c) We first apply the distributive law and then combine like terms.

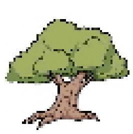
$$P = 2x + 2(3x - 2) = 2x + 6x - 4 = \boxed{8x - 4}.$$

Sometimes expressions are more complicated than the ones we just saw. One complication can be several pairs of parentheses nested inside each other. Just like in case of order of operations, we start with the innermost parentheses.

Example 7. Simplify the expression $-3(2(3x - 1) + 5 - 7x) + 10$

Solution: We will eliminate parentheses by applying the distributive law, combine like terms, and repeat. Note that if we take the time to combine like terms after expanding a multiplication, that will cut down on the computation in the next step.

$$\begin{aligned} -3(2(3x-1)+5-7x)+10 &= && \text{distribute in innermost parentheses} \\ &= -3((6x-2)+5-7x)+10 && \text{drop parentheses, combine like terms} \\ &= -3(-x+3)+10 && \text{apply the distributive law} \\ &= 3x - 9 + 10 && \text{combine like terms} \\ &= \boxed{3x+1} \end{aligned}$$



Enrichment

Think of an integer. Add six to it. Double the sum. Take its opposite. Add eight. Multiply by five. Add ten times your original number. Add 25. Double again. No matter what number you had in mind, the final result is 10. Try this with a few numbers. Can you explain why this happens?

Sample Problems

1. Evaluate each of the following algebrac expressions with the value(s) given.

   a) $-x^2 - 5x + 2$ if $x = -2$ b) $-16t^2 + 32t + 240$ if $t = 3$

2. Add the algebraic expressions as indicated.

   a) $(3x - 5y) + (2x + 4y)$ b) $(2a - 5b + 3) + (-a - 8b + 3)$

3. Multiply the algebraic expressions by a number as indicated.

   a) $3(2x + 4y - 5)$ b) $-5(2a - b + 8)$ c) $-1(-p + 3q - 8m + 6)$ d) $-(-a + 3b - 7)$

4. Subtract the algebraic expressions as indicated.

   a) $(2a + b) - (a - b)$ c) $(2m - 5n + 3) - (-m - 8n + 3)$ e) $(3a - 2) - (1 - 4a)$
   b) $(3x - 5y) - (2x + 4y)$ d) $(2a - 2b) - (b - a)$

5. Simplify each of the following.

   a) $3(x - 5) - 5(x - 1)$ c) $2(a - 2b) + 3(5b - 2a) - 4(2b - a)$
   b) $4(2a - b) - 3(5a - 2b)$ d) $-(3a - 2) - (1 - 4a)$

6. Simplify the given expression. $3(-2(-(6x - 1) + 3(2x - 5) - 5x + 7) - 8x - 10)$

Practice Problems

1. Evaluate each of the following algebrac expressions with the value(s) given.

   a) $3x^2 - x + 7$ if $x = -1$ b) $-a + 5b$ if $a = 3$ and $b = -2$ c) $\dfrac{x^x - 1}{x - 1}$ if $x = 2$

2. Add the algebraic expressions as indicated.

   a) $(2a - 7y + 1) + (2a - 7y - 1)$ c) $(4p - 5q + 1) + (5p - 4q - 2)$ e) $(-5y + 8) + (5y - 8)$
   b) $(-2a - 5) + (3a + 5)$ d) $(2x - 3y + 8) + (-2x - 3y - 8)$

3. Multiply the algebraic expressions by a number as indicated.

   a) $5(-3a + b - 5)$ b) $0(2x - 7y + 8z)$ c) $-6(n - 3m - 8)$ d) $-(p - 5q + 3r - 1)$

4. Subtract the algebraic expressions as indicated.

   a) $(2a - 7y) - (2a - 7y)$ c) $(2x - 5y - 1) - (-y + 1)$
   b) $(-2a - 5b - 1) - (3a + 5b - 1)$ d) $(2m - 3n + 8) - (-2m - 3n - 8)$

5. Simplify each of the following.

   a) $(x + 1) - (x - 1)$ c) $-2 + x - 3(x - 1) - (1 - 2x)$
   b) $2(x - 1) - 3(x - 7)$ d) $-2(a - 3b + c) - 3(a - b + 4c)$

6. Simplify each of the given expressions.

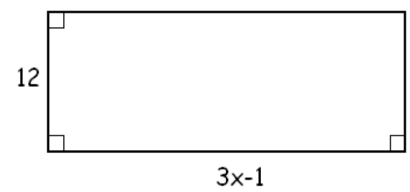
   a) $-2\left(\left(3x-8\left(-2x+1\right)-3x\right)-5x+2\right)$

   b) $3\left(5\left(-2\left(-\left(2x-1\right)+1\right)-2\right)+3\right)-7$

   c) $2\left(\left(5x-1\right)-3\left(\left(8x+1\right)+4\left(x-5\right)-x+1\right)+3x-3\right)$

7. Consider the rectangle shown on the picture. Express the perimeter and area of the rectangle in terms of x. Simplify the expressions.

   Units are not given. This means that the answer can be presented either without units, or perimeter in unit and area in unit^2. (For example, $P=14$ unit and $A=24$ unit^2).



Answers

Sample Problems

1. a) 8 b) 192 2. a) $5x - y$ b) $a - 13b + 6$

3. a) $6x + 12y - 15$ b) $-10a + 5b - 40$ c) $p - 3q + 8m - 6$ d) $a - 3b + 7$

4. a) $a + 2b$ b) $x - 9y$ c) $3m + 3n$ d) $3a - 3b$ e) $7a - 3$

5. a) $-2x - 10$ b) $-7a + 2b$ c) $3b$ d) $a + 1$ 6. $6x + 12$

Practice Problems

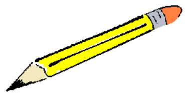
1. a) 11 b) -13 c) 3 2. a) $4a - 14y$ b) a c) $9p - 9q - 1$ d) $-6y$ e) 0

3. a) $5b - 15a - 25$ b) 0 c) $-6n + 18m + 48$ d) $-p + 5q - 3r + 1$

4. a) 0 b) $-5a - 10b$ c) $2x - 4y - 2$ d) $4m + 16$

5. a) 2 b) $-x + 19$ c) 0 d) $-5a + 9b - 14c$ 6. a) $-22x + 12$ b) $60x - 88$ c) $-50x + 100$

7. $P = (6x + 22)$ unit $A = (36x - 12)$ unit^2



Solutions - Sample Problems

1. Evaluate each of the following algebrac expressions with the value(s) given.

a) $-x^2 - 5x + 2$ if $x = -2$

Solution: We substitute -2 into the expression. Please note that if the value of x is negative, we will need to place parentheses around it.

$$-x^2 - 5x + 2 = -(-2)^2 - 5(-2) + 2$$

According to order of operations, we perform the exponentiation first.

$$-(-2)^2 - 5(-2) + 2 = -4 - 5(-2) + 2 = -4 - (-10) + 2 = -4 + 10 + 2 = 6 + 2 = \boxed{8}$$

Why don't the two minuses make a plus in $-(-2)^2$? They do, it's just that there are three minus signs and not two: $-(-2)^2 = -1(-2)(-2)$.

b) $-16t^2 + 32t + 240$ if $t = 3$

Solution: We substitute 3 into the expression. Please note that if the value of x is negative, we will need to place parentheses around it.

$$\begin{aligned} -16t^2 + 32t + 240 &= -16(3)^2 + 32(3) + 240 \\ &= -16 \cdot 9 + 32 \cdot 3 + 240 \\ &= -144 + 96 + 240 = -48 + 240 = \boxed{192} \end{aligned}$$

2. Add the algebraic expressions as indicated.

a) $(3x - 5y) + (2x + 4y)$

Solution: We drop the parentheses and combine like terms.

$$\begin{aligned} (3x - 5y) + (2x + 4y) &= \\ 3x - 5y + 2x + 4y &= && \text{organize like terms together} \\ 3x + 2x - 5y + 4y &= && 3 + 2 = 5 \quad \text{and} \quad -5 + 4 = -1 \\ &= \boxed{5x - y} \end{aligned}$$

b) $(2a - 5b + 3) + (-a - 8b + 3)$

Solution: We drop the parentheses and combine like terms.

$$\begin{aligned} (2a - 5b + 3) + (-a - 8b + 3) &= \\ 2a - 5b + 3 - a - 8b + 3 &= \\ 2a - a - 5b - 8b + 3 + 3 &= && 2 - 1 = 1, \quad -5 - 8 = -13, \quad \text{and} \quad 3 + 3 = 6 \\ 1a - 13b + 6 &= \boxed{a - 13b + 6} \end{aligned}$$

3. Multiply the algebraic expressions by a number as indicated.

a) $3(2x+4y-5)$

Solution: We distribute 3.

$$3(2x+4y-5) = \boxed{6x+12y-15}$$

b) $-5(2a-b+8)$

Solution: We distribute -5.

$$-5(2a-b+8) = \boxed{-10a+5b-40}$$

c) $-1(-p+3q-8m+6)$

Solution: We distribute -1.

$$-1(-p+3q-8m+6) = \boxed{p-3q+8m-6}$$

d) $-(-a+3b-7)$

Solution: The notation here indicates multiplication by -1, which is the same as taking the opposite of a quantity. We distribute -1.

$$-1(-a+3b-7) = \boxed{a-3+7}$$

4. Subtract the algebraic expressions as indicated.

a) $(2a+b)-(a-b)$

Solution: To subtract is to add the opposite. The opposite of $a-b$ is $-a+b$ since

$$-1(a-b) = -a+b$$

Thus
$$\begin{aligned}(2a+b)-(a-b) &= (2a+b)+(-a+b) && \text{drop parentheses}\\ &= 2a+b-a+b && \text{combine like terms}\\ &= \boxed{a+2b}\end{aligned}$$

b) $(3x-5y)-(2x+4y)$

Solution: To subtract is to add the opposite. The opposite of $2x+4y$ is $-2x-4y$ since

$$-1(2x+4y) = -2x-4y$$

$$\begin{aligned}(3x-5y)-(2x+4y) &= (3x-5y)+(-2x-4y) && \text{to subtract is to add the opposite}\\ &= 3x-5y-2x-4y && \text{drop parentheses, combine like terms}\\ &= \boxed{x-9y}\end{aligned}$$

c) $(2m-5n+3)-(-m-8n+3)$

Solution: To subtract is to add the opposite. The opposite of $-m-8n+3$ is $m+8n-3$ since

$$-1(-m-8n+3)=m+8n-3$$

$$\begin{aligned}(2m-5n+3)-(-m-8n+3) &= \quad \text{to subtract is to add the opposite}\\ (2m-5n+3)+(m+8n-3) &= \quad \text{drop parentheses, combine like terms}\\ 2m-5n+3+m+8n-3 &= \boxed{3m+3n}\end{aligned}$$

d) $(2a-2b)-(b-a)$

Solution: To subtract is to add the opposite. The opposite of $b-a$ is $-b+a$ since

$$-1(b-a)=-b+a$$

$$\begin{aligned}(2a-2b)-(b-a) &= (2a-2b)+(-b+a) && \text{to subtract is to add the opposite}\\ &= 2a-2b-b+a && \text{drop parentheses, combine like terms}\\ &= \boxed{3a-3b}\end{aligned}$$

e) $(3a-2)-(1-4a)$

Solution: To subtract is to add the opposite. The opposite of $1-4a$ is $-1+4a$ since

$$-1(1-4a)=-1+4a$$

$$\begin{aligned}(3a-2)-(1-4a) &= \quad \text{to subtract is to add the opposite}\\ (3a-2)+(-1+4a) &= \quad \text{drop parentheses, combine like terms}\\ 3a-2-1+4a &= \boxed{7a-3}\end{aligned}$$

5. Simplify each of the following.

a) $3(x-5)-5(x-1)$

Solution: We apply the law of distributivity and combine like terms. Notice that the last term is $-5(-1)=5$.

$$\begin{aligned}3(x-5)-5(x-1) &= && \text{apply the distributie law}\\ 3x-15-5x+5 &= && \text{combine like terms}\\ &= \boxed{-2x-10}\end{aligned}$$

b) $4(2a-b)-3(5a-2b)$

Solution: We apply the law of distributivity and combine like terms

$$\begin{aligned}4(2a-b)-3(5a-2b) &= 8a-4b-15a+6b\\ &= \boxed{-7a+2b}\end{aligned}$$

c) $2(a-2b)+3(5b-2a)-4(2b-a)$

Solution: We apply the law of distributivity and combine like terms

$$2(a-2b)+3(5b-2a)-4(2b-a) = 2a-4b+15b-6a-8b+4a = \boxed{3b}$$

d) $-(3a-2)-(1-4a)$

Solution:

$$\begin{aligned} -(3a-2)-(1-4a) &= -1(3a-2)-1(1-4a) && \text{multiplication} \\ &= -3a+2-1+4a && \text{combine like terms} \\ &= \boxed{a+1} \end{aligned}$$

6. Simplify the given expression. $3(-2(-(6x-1)+3(2x-5)-5x+7)-8x-10)$

Solution: We first need to investigate the parentheses and pair them off. There is no point starting the computation without understanding them; it will only get worse later. We will denote the different parentheses with differently shaped grouping symbols. There are two innermost parentheses, one after the other. We start there, left to right.

$$\begin{aligned} &3\{-2[-(6x-1)+3(2x-5)-5x+7]-8x-10\} = && \text{distribute } -1 \\ &= 3\{-2[-6x+1+3(2x-5)-5x+7]-8x-10\} && \text{combine like terms} \\ &= 3\{-2[-11x+8+3(2x-5)]-8x-10\} && \text{distribute } 3 \\ &= 3\{-2[-11x+8+6x-15]-8x-10\} && \text{combine like terms} \\ &= 3\{-2(-5x-7)-8x-10\} && \text{distribute } -2 \\ &= 3\{10x+14-8x-10\} && \text{combine like terms} \\ &= 3(2x+4) && \text{distribute } 3 \\ &= \boxed{6x+12} \end{aligned}$$

