Caution! Currently, square roots are defined differently in different textbooks. In conversations about mathematics, approach the concept with caution and first make sure that everyone in the conversation uses the same definition of square roots.

The square of an integer, such as $0, 1, 4, 9, 16, 25, \ldots$ is called a **perfect square**. In this section we will only discuss the square root of perfect squares (and their opposites). The square root of other numbers such as 2, 3, or 10 also exist and is studied later in intermediate algebra. For now, we will be dealing with perfect squares only.

The main idea of square roots is simple. Consider the number 36. What number's square equals to 36? However, there is a complication here. Both 6 and -6, when squared, result in 36. This causes two issues. First, since both 6 and -6 square to 36, no real number, when squared, results in -36. Thus $\sqrt{-36}$ is undefined. This is our second exmple for undefined result, the first being division by zero. The second issue is that mathematicians wanted square root to be an operation, with a unique result. So they defined the square root of 36 to be the *positive* number that when squared, we get 36. So, we define $\sqrt{36}$ to be 6. If we wanted to denote -6, that is not the square root of 36, but rather its opposite, so we will write $-\sqrt{36}$.

> **Definition**: Let N be a non-negative number. Then the **square root of** N (notation: $\sqrt{N}$) is the non-negative number that, if we square, the result is N. If N is negative, then $\sqrt{N}$ is undefined.

For example, $\sqrt{25} = 5$. On the other hand, $\sqrt{-25}$ is undefined. The definition uses the expression non-negative becuse $\sqrt{0}$ exists and is 0.

Example 1. Evaluate each of the given numerical expressions.

a) $\sqrt{49}$ b) $-\sqrt{49}$ c) $\sqrt{-49}$ d) $-\sqrt{-49}$

Solution: a) $\sqrt{49} = \boxed{7}$

b) $-\sqrt{49} = \boxed{-7}$

c) $\sqrt{-49} = \boxed{\text{undefined}}$

d) $-\sqrt{-49} = \boxed{\text{undefined}}$

How do square roots fit into the order of operations agreement? With this new operation, we need to revisit this old topic and see how square roots fit into it. The answer is, **taking the square root is equally strong with exponentiation**, so we perform it before all multiplications, divisions, additions and subtractions. When there are both exponents and square roots in an expression, or several square roots, we perform them left to right.

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Example 2. Evaluate each of the given numerical expressions.

a) $\dfrac{2\sqrt{49} - (-1)^3}{\sqrt{9}}$ b) $\sqrt{100} - 2\left(\sqrt{36} - 3\left(\sqrt{25} - \sqrt{16}\right)\right)$

Solution: a) If we re-write the expression using the division sign instead of the horizontal bar, we get $\left(2\sqrt{49} - (-1)^3\right) \div \sqrt{9}$. Thus we will work out the dividend first. There, we have a multiplication, an exponentiation, and a subtraction. Square root and exponentiation are equally strong, so will perform them left to right.

$$
\begin{aligned}
\frac{2\sqrt{49}-(-1)^3}{\sqrt{9}} &= && \text{square root upstairs}\\
&= \frac{2\cdot 7-(-1)^3}{\sqrt{9}} && \text{exponentiation}\\
&= \frac{2\cdot 7-(-1)}{\sqrt{9}} && \text{multiplication}\\
&= \frac{14-(-1)}{\sqrt{9}} && \text{subtraction}\\
&= \frac{15}{\sqrt{9}} && \text{square root}\\
&= \frac{15}{3} && \text{division}\\
&= \boxed{5}
\end{aligned}
$$

b) There are several pairs of parentheses. We will start with the innermost one.

$$
\begin{aligned}
\sqrt{100}-2\left(\sqrt{36}-3\left(\sqrt{25}-\sqrt{16}\right)\right) & && \text{square roots, left to right}\\
&= \sqrt{100}-2\left(\sqrt{36}-3\left(5-\sqrt{16}\right)\right)\\
&= \sqrt{100}-2\left(\sqrt{36}-3(5-4)\right) && \text{subtraction}\\
&= \sqrt{100}-2\left(\sqrt{36}-3\cdot 1\right) && \text{square root}\\
&= \sqrt{100}-2(6-3\cdot 1) && \text{multiplication}\\
&= \sqrt{100}-2(6-3) && \text{subtraction}\\
&= \sqrt{100}-2\cdot 3 && \text{square root}\\
&= 10-2\cdot 3 && \text{multipliction}\\
&= 10-6 && \text{subtraction}\\
&= \boxed{4}
\end{aligned}
$$

Square roots, when stretched over entire expressions, also serve as grouping symbols. This is what we called an *invisible parentheses*.

Example 3. Evaluate each of the following expressions.

a) $\sqrt{25}-\sqrt{16}$ b) $\sqrt{25-16}$ c) $\sqrt{16}+9$ d) $\sqrt{16+9}$ e) $\sqrt{-\sqrt{100}\div(-2)+\sqrt{(-6)^2-5\sqrt{16}}}$

Solution: a) $\sqrt{25}-\sqrt{16}=5-4=\boxed{1}$

b) $\sqrt{25-16}=\sqrt{9}=\boxed{3}$

These two examples illustrate that the order here matters, so we must be careful not to confuse these two types of expressions.

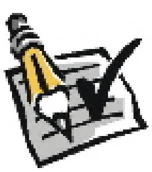
c) $\sqrt{16}+9=4+9=\boxed{13}$

d) $\sqrt{16+9}=\sqrt{25}=\boxed{5}$

These two examples illustrate that we have to be precise with our notation and extend the square root sign over the entire expression as in d) to avoid confusing the two types of expressions.

e) The entire expression is enclosed in a square root, so that will be the last operation to perform. Inside, there is another square root stretched over an entire expression. That serves as parentheses, so we start there. Inside, we start with exponentiations and square roots, left to right.

$$\begin{aligned}
&\sqrt{-\sqrt{100}\div(-2)+\sqrt{(-6)^2-5\sqrt{16}}}= && \text{exponentiation}\\
&=\sqrt{-\sqrt{100}\div(-2)+\sqrt{36-5\sqrt{16}}} && \text{square root}\\
&=\sqrt{-\sqrt{100}\div(-2)+\sqrt{36-5\cdot 4}} && \text{multiplication}\\
&=\sqrt{-\sqrt{100}\div(-2)+\sqrt{36-20}} && \text{subtraction}\\
&=\sqrt{-\sqrt{100}\div(-2)+\sqrt{16}} && \text{square roots, left to right}\\
&=\sqrt{-10\div(-2)+\sqrt{16}} && \\
&=\sqrt{-10\div(-2)+4} && \text{division}\\
&=\sqrt{5+4} && \text{addition}\\
&=\sqrt{9} && \text{square root}\\
&=\boxed{3} &&
\end{aligned}$$



Practice Problems

1. Evaluate each of the following expressions.

   a) $\sqrt{100}$ b) $\sqrt{-100}$ c) $-\sqrt{100}$ d) $-\sqrt{49}$ e) $-\sqrt{-49}$ d) $\sqrt{1}$ e) $\sqrt{0}$

2. Evaluate each of the following expressions.

   a) $\sqrt{\sqrt{81}+\sqrt{49}}$ b) $\sqrt{\sqrt{64}+1}$ c) $\sqrt{\sqrt{11-\sqrt{4}}+\sqrt{1}}$ d) $\sqrt{\sqrt{\sqrt{100}-1}+1}$

3. Evaluate each of the following expressions.

   a) $\sqrt{25}-2\sqrt{9-(-3)^3}=-7$ b) $\sqrt{\sqrt{36}+5\sqrt{9}-\dfrac{\sqrt{100}}{2}}$ c) $\sqrt{\sqrt{4\sqrt{64}-\sqrt{49}}-\sqrt{1}}$

4. Evaluate each of the following expressions.

   a) $\sqrt{16-2\sqrt{-4^2-\sqrt{1}+2\cdot 3^2\div\sqrt{4}\cdot 6-1}}$ b) $\sqrt{3\sqrt{25}-\left(2\sqrt{100}-5\sqrt{16}-\sqrt{36}\div 6\right)}$

Answers

1. a) 10 b) undefined c) -10 d) -7 e) undefined d) 1 e) 0
2. a) 4 b) 3 c) 2 d) 2
3. a) -7 b) 4 c) 2
4. a) 2 b) 4

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