We already know how to perform operations with radical expressions. We can add, subtract, multiply them. The only thing missing is division. Dividing by a rational number is just a matter of the distributive law.

Example 1. Perform the division $\dfrac{2\sqrt{5}-4}{7}$.

Solution: To divide is to multiply by the reciprocal.

$$\frac{2\sqrt{5}-4}{7} = \left(2\sqrt{5}-4\right)\cdot\frac{1}{7} = \boxed{\frac{2}{7}\sqrt{5}-\frac{4}{7}}$$

But how do we divide by an expression that contains radicals? Recall that if we multiply two expressions that are conjugates, the result is rational.

Example 2. Perform the given multiplications.

a) $(3\sqrt{7}-5)(3\sqrt{7}+5)$ b) $(\sqrt{10}+3)(\sqrt{10}-3)$

Solution: a) We apply the distributive law. Because we are multiplying conjugates, the terms O and I from FOIL cancel out each other and we are left with the difference of two squares.

$$\left(3\sqrt{7}-5\right)\left(3\sqrt{7}+5\right) = \left(3\sqrt{7}\right)^2 + 5\left(3\sqrt{7}\right) - 5\left(3\sqrt{7}\right) - 5^2 = 9\cdot 7 - 25 = 63 - 25 = \boxed{38}$$

b) $(\sqrt{10}+3)(\sqrt{10}-3) = \left(\sqrt{10}\right)^2 - 3\sqrt{10} + 3\sqrt{10} - 3^2 = 10 - 9 = \boxed{1}$

In this last example, the two numbers are not only conjugates, but they are also reciprocals of each other.

$$\left(\sqrt{10}+3\right)\left(\sqrt{10}-3\right) = 1 \quad\Longrightarrow\quad \sqrt{10}+3 = \frac{1}{\sqrt{10}-3}$$

We should be able to compute the reciprocal of $\sqrt{10}-3$ directly, instead of just running into it by chance. Indeed, we can transform quotients so that their denominators become rational. This is also called rationalizing the denominator.

Example 3. Rationalize the denominator in each of the given expressions.

a) $\dfrac{6}{2\sqrt{7}-5}$ b) $\dfrac{2}{\sqrt{5}-1}$ c) $\dfrac{\sqrt{3}+5}{\sqrt{3}-1}$

Solution: a) We will multiply the expression by 1, where 1 is written as a fraction with the same numerator and denominator. That same expression is the conjugate of the part to be rationalized. In case of $2\sqrt{7}-5$, the conjugate is $2\sqrt{7}+5$.

$$\begin{aligned}\frac{6}{2\sqrt{7}-5} &= \frac{6}{2\sqrt{7}-5}\cdot 1 = \frac{6}{2\sqrt{7}-5}\cdot\frac{2\sqrt{7}+5}{2\sqrt{7}+5} = \frac{6\left(2\sqrt{7}+5\right)}{(2\sqrt{7}-5)(2\sqrt{7}+5)} = \frac{6\left(2\sqrt{7}+5\right)}{\left(2\sqrt{7}\right)^2-5^2}\\ &= \frac{6\left(2\sqrt{7}+5\right)}{28-25} = \frac{6\left(2\sqrt{7}+5\right)}{28-25} = \frac{6\left(2\sqrt{7}+5\right)}{3} = \frac{2\left(2\sqrt{7}+5\right)}{1} = \boxed{4\sqrt{7}+10}\end{aligned}$$

b) $$\frac{2}{\sqrt{5}-1} = \frac{2}{\sqrt{5}-1}\cdot 1 = \frac{2}{\sqrt{5}-1}\cdot\frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{2\left(\sqrt{5}+1\right)}{\left(\sqrt{5}\right)^2-1^2} = \frac{2\left(\sqrt{5}+1\right)}{5-1} = \frac{2\left(\sqrt{5}+1\right)}{4} = \boxed{\frac{\sqrt{5}+1}{2}}$$

c) $$\begin{aligned}\frac{\sqrt{3}+5}{\sqrt{3}-1} &= \frac{\sqrt{3}+5}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{(\sqrt{3}+5)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{\left(\sqrt{3}\right)^2+\sqrt{3}+5\sqrt{3}+5}{\left(\sqrt{3}\right)^2-1^2} = \frac{3+6\sqrt{3}+5}{3-1}\\ &= \frac{8+6\sqrt{3}}{2} = \frac{2\left(4+6\sqrt{3}\right)}{2} = \boxed{4+3\sqrt{3}}\end{aligned}$$

Sample Problems

1. Rationalize the denominator in each of the following expressions.

   a) $\dfrac{3}{\sqrt{5}}$ b) $\dfrac{1}{\sqrt{10}-3}$ c) $\dfrac{2}{\sqrt{7}+1}$ d) $\dfrac{\sqrt{5}-1}{\sqrt{5}-2}$ e) $\dfrac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}$

2. Simplify each of the following expressions.

   a) $\dfrac{3-\sqrt{5}}{3+\sqrt{5}}+\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$

   b) $\left(\dfrac{8}{\sqrt{7}+\sqrt{3}}+\dfrac{12}{\sqrt{7}-\sqrt{3}}\right)\left(5\sqrt{7}-\sqrt{3}\right)$

   c) $\sqrt{\sqrt{41}+4\sqrt{2}}\cdot\sqrt{\sqrt{41}-\sqrt{32}}$

   d) $\sqrt{5\sqrt{3}+\sqrt{59}}\cdot\sqrt{\sqrt{75}-\sqrt{59}}$

   e) $\left(\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}\right)^2$

3. Find the exact value of x^2-4x+6 if $x=2-\sqrt{3}$.

4. Solve each of the following quadratic equations by completing the square. Check your solution(s).

   a) $x^2=4x+1$ b) $x^2+13=8x$

Practice Problems

1. Rationalize the denominator in each of the following expressions.

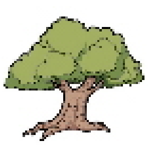
   a) $\dfrac{4}{\sqrt{7}}$ b) $\dfrac{1}{\sqrt{7}-3}$ c) $\dfrac{1}{\sqrt{10}+3}$ d) $\dfrac{2}{\sqrt{17}+4}$ e) $\dfrac{\sqrt{5}-1}{\sqrt{5}+1}$ f) $\dfrac{\sqrt{10}+\sqrt{2}}{\sqrt{10}-\sqrt{2}}$ g) $\dfrac{-5x+5}{\sqrt{x}+1}$ h) $\dfrac{2}{\sqrt{x}+4}$

2. Find the exact value of

   a) x^2-6x+1 if $x=3-\sqrt{10}$ b) $b^2+8b-20$ if $b=\sqrt{5}-2$ c) $x^2-10x+16$ if $x=\sqrt{6}+5$

3. Solve each of the following quadratic equations by completing the square. Check your solution(s).

   a) $x^2+47=14x$ b) $x^2+12x+31=0$ c) $x^2=2x+1$



Enrichment

Simplify each of the following expressions.

1. $\dfrac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}}$ 2. $\dfrac{\sqrt{1008+\sqrt{2015}}-\sqrt{1008-\sqrt{2015}}}{\sqrt{2}}$

Answers

Sample Problems

1. a) $4\sqrt{2}$ b) $3\sqrt{5}$ c) $4x^2y\sqrt{3xy}$ d) $-4\sqrt{5}$ e) $\dfrac{2}{3}$ f) $13a^5\sqrt{5a}$ g) 3 h) $11-4\sqrt{7}$
   i) $-10+6\sqrt{3}$ j) $5x+\sqrt{5x}-6$ k) $6+\sqrt{x}-2x$ l) $x-2\sqrt{2x}+2$
2. a) $\dfrac{3\sqrt{5}}{5}$ b) $3+\sqrt{10}$ c) $\dfrac{-1+\sqrt{7}}{3}$ d) $3+\sqrt{5}$ e) $\dfrac{13-4\sqrt{10}}{3}$
3. a) 7 b) 172 c) 3 d) 4 e) 22 4. 5 5. a) $2-\sqrt{5},\ 2+\sqrt{5}$ b) $4-\sqrt{3},\ 4+\sqrt{3}$

Practice Problems

1. a) $2\sqrt{5}$ b) $2\sqrt{3}$ c) $8\sqrt{5}$ d) $7a^2\sqrt{2a}$ e) 1 f) $9-4\sqrt{5}$ g) $17\sqrt{5}-38$ h) $103-\sqrt{5}$
2. a) $\dfrac{4\sqrt{7}}{7}$ b) $-\dfrac{3+\sqrt{7}}{2}$ c) $-3+\sqrt{10}$ d) $-8+2\sqrt{17}$ or $2\left(-4+\sqrt{17}\right)$ e) $\dfrac{3-\sqrt{5}}{2}$ f) $\dfrac{\sqrt{5}+3}{2}$
   g) $-5\sqrt{x}+5$ h) $\dfrac{2\sqrt{x}-8}{x-16}$ 3. a) 2 b) $-27+4\sqrt{5}$ c) -3
4. a) $7-\sqrt{2},\ 7+\sqrt{2}$ b) $-6-\sqrt{5},\ -6+\sqrt{5}$ c) $1-\sqrt{2},\ 1+\sqrt{2}$

Sample Problems Solutions

1. Rationalize the denominator in each of the following expressions.

   a) $\dfrac{3}{\sqrt{5}}$

   Solution: To rationalize the denominator, we will multiply both the numerator and denominator by $\sqrt{5}$.

$$\frac{3}{\sqrt{5}}=\frac{3}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\boxed{\frac{3\sqrt{5}}{5}}$$

   b) $\dfrac{1}{\sqrt{10}-3}$

   Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is $\sqrt{10}+3$.

$$\frac{1}{\sqrt{10}-3}=\frac{1}{\sqrt{10}-3}\cdot\frac{\sqrt{10}+3}{\sqrt{10}+3}=\frac{\sqrt{10}+3}{1}=\boxed{3+\sqrt{10}}$$

The denominator is 1 since

$$\left(\sqrt{10}-3\right)\left(\sqrt{10}+3\right)=\sqrt{10}\sqrt{10}+3\sqrt{10}-3\sqrt{10}-9=10-9=1$$

c) $\dfrac{2}{\sqrt{7}+1}$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is $\sqrt{7}-1$

$$\frac{2}{\sqrt{7}+1}=\frac{2}{\sqrt{7}+1}\cdot\frac{\sqrt{7}-1}{\sqrt{7}-1}=\frac{2\left(\sqrt{7}-1\right)}{7-1}=\frac{2\left(\sqrt{7}-1\right)}{6}=\boxed{\frac{\sqrt{7}-1}{3}}$$

d) $\dfrac{\sqrt{5}-1}{\sqrt{5}-2}$

Solution:

$$\frac{\sqrt{5}-1}{\sqrt{5}-2}\cdot\frac{\sqrt{5}+2}{\sqrt{5}+2}=\frac{\left(\sqrt{5}-1\right)\left(\sqrt{5}+2\right)}{\left(\sqrt{5}-2\right)\left(\sqrt{5}+2\right)}=\frac{5+2\sqrt{5}-\sqrt{5}-2}{5+2\sqrt{5}-2\sqrt{5}-4}=\frac{3+\sqrt{5}}{1}=\boxed{3+\sqrt{5}}$$

e) $\dfrac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}$

Solution: We multiply the fraction by 1 as a fraction of whose both numerator and denominator are the conjugate of the denominator.

$$\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}=\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}\cdot 1=\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}\cdot\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}-\sqrt{5}}=\frac{\left(\sqrt{8}-\sqrt{5}\right)\left(\sqrt{8}-\sqrt{5}\right)}{\left(\sqrt{8}+\sqrt{5}\right)\left(\sqrt{8}-\sqrt{5}\right)}$$

We FOIL out both numerator and denominator

$$\frac{\sqrt{8}\sqrt{8}-\sqrt{8}\sqrt{5}-\sqrt{5}\sqrt{8}+\sqrt{5}\sqrt{5}}{\sqrt{8}\sqrt{8}-\sqrt{8}\sqrt{5}+\sqrt{5}\sqrt{8}-\sqrt{5}\sqrt{5}}=\frac{8-\sqrt{40}-\sqrt{40}+5}{8-5}=\frac{13-2\sqrt{40}}{3}$$

Note: although this answer is acceptable, the expression can be further simplified.

$$\frac{13-2\sqrt{40}}{3}=\frac{13-2\sqrt{4\cdot 10}}{3}=\frac{13-2\sqrt{4}\cdot\sqrt{10}}{3}=\frac{13-2\cdot 2\cdot\sqrt{10}}{3}=\boxed{\frac{13-4\sqrt{10}}{3}}$$

2. Simplify each of the following expressions.

a) $\dfrac{3-\sqrt{5}}{3+\sqrt{5}}+\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$

Solution:

$$\frac{3-\sqrt{5}}{3+\sqrt{5}}+\frac{3+\sqrt{5}}{3-\sqrt{5}}=\frac{\left(3-\sqrt{5}\right)^2+\left(3+\sqrt{5}\right)^2}{\left(3+\sqrt{5}\right)\left(3-\sqrt{5}\right)}=\frac{14-6\sqrt{5}+14+6\sqrt{5}}{9-5}=\frac{28}{4}=\boxed{7}$$

b) $\left(\dfrac{8}{\sqrt{7}+\sqrt{3}}+\dfrac{12}{\sqrt{7}-\sqrt{3}}\right)\left(5\sqrt{7}-\sqrt{3}\right)$

Solution: We first bring the fractions to the common denominator - they are conjugates, that helps, because the common denominator is then rational. We then multiply the factors.

$$\left(\frac{8}{\sqrt{7}+\sqrt{3}}+\frac{12}{\sqrt{7}-\sqrt{3}}\right)\left(5\sqrt{7}-\sqrt{3}\right)=$$

$$\begin{aligned}
&= \frac{8\left(\sqrt{7}-\sqrt{3}\right)+12\left(\sqrt{7}+\sqrt{3}\right)}{\left(\sqrt{7}+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{3}\right)}\left(5\sqrt{7}-\sqrt{3}\right)\\
&= \frac{8\sqrt{7}-8\sqrt{3}+12\sqrt{7}+12\sqrt{3}}{7-3}\left(5\sqrt{7}-\sqrt{3}\right)\\
&= \frac{20\sqrt{7}+4\sqrt{3}}{4}\left(5\sqrt{7}-\sqrt{3}\right)=\left(5\sqrt{7}+\sqrt{3}\right)\left(5\sqrt{7}-\sqrt{3}\right)\\
&= 25\,(7)-3=175-3=\boxed{172}
\end{aligned}$$

c) $\sqrt{\sqrt{41}+4\sqrt{2}}\cdot\sqrt{\sqrt{41}-\sqrt{32}}$

Solution:

$$\begin{aligned}
\sqrt{\sqrt{41}+4\sqrt{2}}\cdot\sqrt{\sqrt{41}-\sqrt{32}} &= \sqrt{\sqrt{41}+\sqrt{32}}\cdot\sqrt{\sqrt{41}-\sqrt{32}}=\sqrt{\left(\sqrt{41}+\sqrt{32}\right)\left(\sqrt{41}-\sqrt{32}\right)}\\
&= \sqrt{\left(\left(\sqrt{41}\right)^2-\left(\sqrt{32}\right)^2\right)}=\sqrt{41-32}=\sqrt{9}=\boxed{3}
\end{aligned}$$

d) $\sqrt{5\sqrt{3}+\sqrt{59}}\cdot\sqrt{\sqrt{75}-\sqrt{59}}$

Solution:

$$\begin{aligned}
\sqrt{5\sqrt{3}+\sqrt{59}}\cdot\sqrt{\sqrt{75}-\sqrt{59}} &= \sqrt{\sqrt{75}+\sqrt{59}}\cdot\sqrt{\sqrt{75}-\sqrt{59}}=\sqrt{\left(\sqrt{75}+\sqrt{59}\right)\left(\sqrt{75}-\sqrt{59}\right)}\\
&= \sqrt{\left(\left(\sqrt{75}\right)^2-\left(\sqrt{59}\right)^2\right)}=\sqrt{75-59}=\sqrt{16}=\boxed{4}
\end{aligned}$$

e) $\left(\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}\right)^2$

Solution:

$$\begin{aligned}
\left(\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}\right)^2 &= \left(\sqrt{6+\sqrt{11}}\right)^2+\left(\sqrt{6-\sqrt{11}}\right)^2+2\sqrt{6+\sqrt{11}}\sqrt{6-\sqrt{11}}\\
&= 6+\sqrt{11}+6-\sqrt{11}+2\sqrt{\left(6+\sqrt{11}\right)\left(6-\sqrt{11}\right)}\\
&= 12+2\sqrt{36-11}=12+2\sqrt{25}=12+2\cdot 5=\boxed{22}
\end{aligned}$$

3. Find the exact value of x^2-4x+6 if $x=2-\sqrt{3}$.

Solution: We work out x^2 first.

$$\begin{aligned}
x^2 &= \left(2-\sqrt{3}\right)^2=\left(2-\sqrt{3}\right)\left(2-\sqrt{3}\right)\\
&= 4-2\sqrt{3}-2\sqrt{3}+\sqrt{3}\sqrt{3}\\
&= 4-4\sqrt{3}+3=7-4\sqrt{3}
\end{aligned}$$

Now we substitute $x=2-\sqrt{3}$ into x^2-4x+6.

$$\begin{aligned}
x^2-4x+6 &= \left(2-\sqrt{3}\right)^2-4\left(2-\sqrt{3}\right)+6=\\
&= 7-4\sqrt{3}-8+4\sqrt{3}+6=7-8+6=\boxed{5}
\end{aligned}$$

4. Solve each of the following quadratic equations by completing the square. Check your solution(s).

a) $x^2 = 4x + 1$

Solution: We factor by completing the square.

$$\begin{aligned} x^2 &= 4x+1 && \text{reduce one side to zero} \\ x^2-4x-1 &= 0 && (x-2)^2 = x^2-4x+4 \\ \underbrace{x^2-4x+4}-4-1 &= 0 \\ (x-2)^2-5 &= 0 \\ (x-2)^2-\left(\sqrt{5}\right)^2 &= 0 \\ \left(x-2+\sqrt{5}\right)\left(x-2-\sqrt{5}\right) &= 0 \end{aligned}$$

$$\boxed{x_1 = 2-\sqrt{5} \quad \text{and} \quad x_2 = 2+\sqrt{5}}$$

We check: if $x = 2-\sqrt{5}$, then

$$\begin{aligned} \text{LHS} &= \left(2-\sqrt{5}\right)^2 = \left(2-\sqrt{5}\right)\left(2-\sqrt{5}\right) = 4-2\sqrt{5}-2\sqrt{5}+\sqrt{5}\sqrt{5} = 4-4\sqrt{5}+5 = 9-4\sqrt{5} \\ \text{RHS} &= 4\left(2-\sqrt{5}\right)+1 = 8-4\sqrt{5}+1 = 9-4\sqrt{5} \end{aligned}$$

and if $x = 2+\sqrt{5}$, then

$$\begin{aligned} \text{LHS} &= \left(2+\sqrt{5}\right)^2 = \left(2+\sqrt{5}\right)\left(2+\sqrt{5}\right) = 4+2\sqrt{5}+2\sqrt{5}+\sqrt{5}\sqrt{5} = 4+4\sqrt{5}+5 = 9+4\sqrt{5} \\ \text{RHS} &= 4\left(2+\sqrt{5}\right)+1 = 8+4\sqrt{5}+1 = 9+4\sqrt{5} \end{aligned}$$

So, both solutions are correct.

b) $x^2 + 13 = 8x$

Solution: We factor by completing the square.

$$\begin{aligned} x^2-8x+13 &= 0 && (x-4)^2 = x^2-8x+16 \\ \underbrace{x^2-8x+16}-16+13 &= 0 \\ (x-4)^2-3 &= 0 \\ (x-4)^2-\left(\sqrt{3}\right)^2 &= 0 \\ \left(x-4+\sqrt{3}\right)\left(x-4-\sqrt{3}\right) &= 0 \end{aligned}$$

$$\boxed{x_1 = 4-\sqrt{3} \quad \text{and} \quad x_2 = 4+\sqrt{3}}$$

We check: if $x = 4-\sqrt{3}$, then

$$\begin{aligned} \text{LHS} &= \left(4-\sqrt{3}\right)^2+13 = \left(4-\sqrt{3}\right)\left(4-\sqrt{3}\right)+13 = 16-4\sqrt{3}-4\sqrt{3}+3+13 = 32-8\sqrt{3} \\ \text{RHS} &= 8\left(4-\sqrt{3}\right) = 32-8\sqrt{3} \end{aligned}$$

and if $x = 4+\sqrt{3}$, then

$$\begin{aligned} \text{LHS} &= \left(4+\sqrt{3}\right)^2 + 13 = \left(4+\sqrt{3}\right)\left(4+\sqrt{3}\right) + 13 = 16 + 4\sqrt{3} + 4\sqrt{3} + 3 + 13 = 32 + 8\sqrt{3} \\ \text{RHS} &= 8\left(4+\sqrt{3}\right) = 32 + 8\sqrt{3} \end{aligned}$$

So, both solutions are correct.

