Part 1 - When the leading coefficient is 1

Factoring by completing the square is an extremely powerful factoring technique. We will see later that this is the only method that does not break down once numbers stop being "nice". Recall that a **complete square** or a **perfect square** is the square of a sum or a difference. For example, the expressions $(2a-3)^2$ and $(x+7)^2$ are complete (or perfect) squares.

Example 1. Factor $-4x+x^2-21$ by completing the square.

Step 1. We re-arrange the terms by decreasing order of degree.

$$-4x+x^2-21=x^2-4x-21$$

Step 2A. We obtain the "magic number", that is half of the linear coefficient. The linear coefficient is the number multiplying x, **sign included**. In our example, the magic number is $\dfrac{-4}{2}=-2$. We do not write this line in the main computation.

Step 2B. We place an x in front of the magic number, and square the expression we obtained. Work out this computation on the margin, not in the main computation.

$$(x-2)^2=(x-2)(x-2)=x^2-2x-2x+4=x^2-4x+4$$

Step 2C. We write the "helper line" $(x-2)^2=x^2-4x+4$ in the upper right hand side of the paper. We will use it twice. Our computation so far looks like this:

$$-4x+x^2-21=x^2-4x-21 \qquad\qquad (x-2)^2=x^2-4x\,\boxed{+4}$$

Step 3. The smuggling step. What we have achieved in Step 2, is to have found the only perfect square that begins with the same two terms, x^2-4x as our expression to be factored. We can see that the last term, $+4$ is missing. We complete the square as follows.

Step 3A. Write down our expression with one modification: we leave a gap between the second and third terms.

$$x^2-4x \qquad\quad -21$$

Step 3B. We add zero to the expression by adding and then immediately subtracting 4 into the gap.

$$x^2-4x\;+4-4-21$$

Step 4. We have obtained five terms. We re-write the first three terms as a perfect square (the second time we used the helper line) and combine the last two terms.

$$x^2-4x-21=\underbrace{x^2-4x+4}-4-21=(x-2)^2-25$$

Step 5. We re-write the last number as a square.

$$(x-2)^2-25=(x-2)^2-5^2$$

Step 6. If applies, we factor via the difference of squares theorem.

$$(x-2)^2-5^2=(x-2+5)(x-2-5)$$

Step 7. (Cleanup) We simplify the factors by combining like terms.

$$(x-2+5)(x-2-5)=\boxed{(x+3)(x-7)}$$

Step 8. We check our result by multiplication.

$$(x+3)(x-7) = x^2 - 7x + 3x + 21 = x^2 - 4x + 21$$

Thus our result, $(x+3)(x-7)$ is correct.

The entire computation should look like this:

$$-4x + x^2 - 21 =$$

$$(x-2)^2 = x^2 - 4x \boxed{+4}$$

$$\begin{aligned} &= x^2 - 4x - 21 \\ &= \underbrace{x^2 - 4x + 4} - 4 - 21 \\ &= (x-2)^2 - 25 \\ &= (x-2)^2 - 5^2 \\ &= (x-2+5)(x-2-5) \\ &= \boxed{(x+3)(x-7)} \end{aligned}$$

We check: $(x+3)(x-7) = x^2 - 7x + 3x - 21 = x^2 - 4x - 21$

Example 2. Factor $32 - 18x + x^2$ by completing the square.

Step 1. We re-arrange the terms by decreasing order of degree.

$$32 - 18x + x^2 = x^2 - 18x + 32$$

Step 2A. We obtain the "magic number", that is half of the linear coefficient. (The linear coefficient is the number multiplying x, sign included.) In our example, the magic number is $\dfrac{-18}{2} = -9$.

Step 2B. We place an x in front of the magic number, and square the expression we obtained. Do not write this computation in the main computation.

$$(x-9)^2 = (x-9)(x-9) = x^2 - 9x - 9x + 81 = x^2 - 18x + 81$$

Step 2C. We write the "helper line" $(x-9)^2 = x^2 - 18x + 81$ in the upper right hand side of the paper.

$$x^2 - 18x + 32 \qquad\qquad (x-9)^2 = x^2 - 18x + \boxed{81}$$

Step 3. (The smuggling step.) What we have achieved in Step 3, is to have found the only perfect square that begins with the same two terms, $x^2 - 18x$ as our expression to be factored. We can see that the last term, $+81$ is missing. We complete the square by writing down our expression with a gap between the second and third terms, and then adding zero to the expression by adding and then immediately subtracting 81 in the gap.

$$x^2 - 18x + 32 = x^2 - 18x + 81 - 81 + 32$$

Step 4. We obtained five terms. We re-write the first three terms as a perfect square and combine the last two terms.

$$x^2 - 18x + 32 = \underbrace{x^2 - 18x + 81} - 81 + 32 = (x-9)^2 - 49$$

Step 5. We re-write the last number as a square.

$$(x-9)^2 - 49 = (x-9)^2 - 7^2$$

Step 6. If applies, we factor via the difference of squares theorem.

$$(x-9)^2 - 7^2 = (x-9+7)(x-9-7)$$

Step 7. (Cleanup) We simplify the factors by combining like terms.

$$(x-9+7)(x-9-7) = (x-2)(x-16)$$

Step 8. We check back by multiplication. (See below.)

The entire computation should look like this:

$$32 - 18x + x^2 =$$

$$\begin{aligned} &= x^2 - 18x + 32 \qquad\qquad (x-9)^2 = x^2 - 18x \boxed{+81} \\ &= \underbrace{x^2 - 18x + 81} - 81 + 32 \\ &= (x-9)^2 - 49 \\ &= (x-9)^2 - 7^2 \\ &= (x-9+7)(x-9-7) \\ &= \boxed{(x-2)(x-16)} \end{aligned}$$

We check: $(x-2)(x-16) = x^2 - 2x - 16x + 32 = x^2 - 18x + 32$

Thus our result, $(x-2)(x-16)$ is correct.

Example 3. Factor $28x + x^2 - 1173$ by completing the square.

We first rearrange the terms by degrees.

$28x + x^2 - 1173 =$

$= x^2 + 28x - 1173$ half of the linear coefficient is $\dfrac{28}{2} = 14$

We work out $(x + 14)^2 = x^2 + 28x + 196$ on the margin. So we know to smuggle in 196.

$$\begin{aligned}
&= x^2 + 28x - 1173 && (x+14)^2 = x^2 + 28x \boxed{+196} \\
&= \underbrace{x^2 + 28x + 196} - 196 - 1173 && \text{realize comple square, combine like terms} \\
&= (x+14)^2 - 1369 && \text{from calculator, } \sqrt{1369} = 37 \\
&= (x+14)^2 - 37^2 && \text{difference of squares theorem} \\
&= (x + 14 + 37)(x + 14 - 37) && \text{combine like terms} \\
&= \boxed{(x+51)(x-23)}
\end{aligned}$$

We check by multiplication: $(x + 51)(x - 23) = x^2 - 23x + 51x - 1173 = x^2 + 28x - 1173$

Thus our result, $(x + 51)(x - 23)$ is correct.

Example 4. Factor $17 - 2a + a^2$ by completing the square.

Solution: We first rearrange the terms by degree.

$17 - 2a + a^2 =$

$$\begin{aligned}
&= 17 - 2a + a^2 && \text{rearrange terms} \\
&= a^2 - 2a + 17 && \text{the "magic number" is } \frac{-2}{2} = -1
\end{aligned}$$

We work out $(a - 1)^2 = a^2 - 2a + 1$ on the margin.

$$\begin{aligned}
&= a^2 - 2a + 17 && (a-1)^2 = a^2 - 2a \boxed{+1}\text{, so we smuggle in 1} \\
&= \underbrace{a^2 - 2a + 1} - 1 + 17 && \text{realize comple square, combine like terms} \\
&= (a-1)^2 + 16
\end{aligned}$$

We can not apply the difference of squares theorem, since 16 is added, not subtracted. **The sum of squares can not be factored**, and so the expression $a^2 - 2a + 17$ can not be factored.

Practice Problems

Factor each of the following by completing the square.

1. $x^2 - 10x + 21$
2. $x^2 - 6x + 8$
3. $22y + y^2 + 105$
4. $b^2 - 4b - 45$
5. $14a + a^2 - 51$
6. $b^2 - 10b + 26$
7. $3 + x^2 - 4x$
8. $d^2 + 2d + 2$
9. $6x + x^2 - 432$
10. $x^2 - 14x + 58$
11. $m^2 - 42m + 432$
12. $x^2 - 50x + 525$
13. $10y + y^2 - 375$
14. $x^2 - 40x + 336$
15. $x^2 - 6x + 25$
16. $q^2 - 2q - 48$
17. $x^2 - 18x + 81$
18. $t^2 - 36t - 4437$
19. $x^2 - 46x + 360$
20. $14q + q^2 - 2352$

Answers

1. $(x-3)(x-7)$ 2. $(x-2)(x-4)$ 3. $(y+15)(y+7)$ 4. $(b+5)(b-9)$ 5. $(a+17)(a-3)$

6. can not be factored 7. $(x-1)(x-3)$ 8. can not be factored 9. $(x+24)(x-18)$

10. can not be factored 11. $(m-18)(m-24)$ 12. $(x-15)(x-35)$ 13. $(y+25)(y-15)$

14. $(x-12)(x-28)$ 15. can not be factored 16. $(q+6)(q-8)$ 17. $(x-9)^2$

18. $(t+51)(t-87)$ 19. $(x-10)(x-36)$ 20.) $(q+56)(q-42)$

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