Factoring by completing the square is an extremely powerful factoring technique. We will see later that this is the only method that does not break down once numbers stop being "nice".

Example 1. Factor $18x - 3x^2 + 165$ by completing the square.

Step 1. We re-arrange the terms by decreasing order of degree.

$$18x - 3x^2 + 165 = -3x^2 + 18x + 165$$

Step 2. We factor out the greatest common factor.

$$-3x^2 + 18x + 165 = -3\left(x^2 - 6x - 55\right)$$

Step 3. We factor the expression within the parentheses by completing the square.

$$(x-3)^2 = x^2 - 6x + 9$$

$$\begin{aligned} -3\left(x^2 - 6x - 55\right) &= \\ &= -3\left(\underbrace{x^2 - 6x + 9} - 9 - 55\right) \\ &= -3\left((x-3)^2 - 64\right) \\ &= -3\left((x-3)^2 - 8^2\right) \\ &= -3(x-3+8)(x-3-8) = \boxed{-3(x+5)(x-11)} \end{aligned}$$

Step 4. We check our result by multiplication.

$$-3(x+5)(x-11) = -3\left(x^2 - 11x + 5x - 55\right) = -3\left(x^2 - 6x - 55\right) = -3x^2 + 18x + 165$$

Thus our result, $-3(x+5)(x-11)$ is correct.

Example 2. Factor $267x^2 - 48x^3 + 3x^4$ by completing the square.

$$\begin{aligned} &= 267x^2 - 48x^3 + 3x^4 && \text{rearrange terms} \\ &= 3x^4 - 48x^3 + 267x^2 && \text{factor out } 3x^2 \\ &= 3x^2\left(x^2 - 16x + 89\right) && (x-8)^2 = x^2 - 16x \boxed{+64} \\ &= 3x^2\left(\underbrace{x^2 - 16x + 64} - 64 + 89\right) = && \text{realize comple square, combine like terms} \\ &= 3x^2\left((x-8)^2 + 25\right) \end{aligned}$$

We can not apply the difference of squares theorem, since 25 is added, not subtracted. The sum of squares does not factor. Thus the expression $\boxed{3x^2\left(x^2 - 16x + 89\right)}$ is completely factored.

Example 3. Factor $5x^2 - 240x + 2160$ by completing the square.

$5x^2 - 240x + 2160 =$ factor out 5

$$\begin{aligned}
&= 5\left(x^2 - 48x + 432\right) \\
&= 5\left(\underbrace{x^2 - 48x + 576} - 576 + 432\right) \\
&= 5\left((x-24)^2 - 144\right) \\
&= 5\left((x-24)^2 - 12^2\right) \\
&= 5(x - 24 + 12)(x - 24 - 12) \\
&= \boxed{5(x-12)(x-36)}
\end{aligned}$$

$$(x-24)^2 = x^2 - 48x \boxed{+576}$$

We check: $5(x-12)(x-36) = 5\left(x^2 - 12x - 36x + 432\right) = 5\left(x^2 - 48x + 432\right) = 5x^2 - 240x + 2160$.
Thus our result, $5(x-12)(x-36)$ is correct.

Example 4. Factor $9 - y^2 - 8y$ by completing the square.
We first rearrange the terms be degree and then factor out the leading coefficient.

$9 - y^2 - 8y =$

$$(y+4)^2 = y^2 + 8y \boxed{+16}$$

$$\begin{aligned}
&= -y^2 - 8y + 9 \\
&= -1\left(y^2 + 8y - 9\right) \\
&= -\left(\underbrace{y^2 + 8y + 16} - 16 - 9\right) && \text{realize comple square, combine like terms} \\
&= -\left((y+4)^2 - 25\right) && \text{re-write 25 as a square} \\
&= -\left((y+4)^2 - 5^2\right) && \text{factor via the difference of squares theorem} \\
&= -(y + 4 + 5)(y + 4 - 5) && \text{combine like terms, drop extra parentheses} \\
&= \boxed{-(y+9)(y-1)}
\end{aligned}$$

We check: $-(y+9)(y-1) = -\left(y^2 - y + 9y - 9\right) = -\left(y^2 + 8y - 9\right) = -y^2 - 8y + 9$
Thus our result, $-(y+9)(y-1)$ is correct.

Practice Problems

Completely factor each of the following by completing the square.

1. $4x + 2x^2 - 30$
2. $70a^2 - 255a + 5a^3$
3. $78b^2 - 30b^3 + 3b^4$
4. $32x + 2x^2 - 594$
5. $18c - 24c^2 + 6c^3$
6. $-2d - 2d^2 - d^3$
7. $432 - x^2 - 6x$
8. $x^2 - 14x + 58$
9. $10abc - 600ac + 5ab^2c$
10. $70y^3 + 24y^4 + 2y^5$
11. $18x^2y^2 - 216x^2y + 3x^2y^3$
12. $1000x - 50x^2 - 5x^3$

Answers

1. $2(x+5)(x-3)$ 2. $5a(a-3)(a+17)$ 3. $3b^2(b^2-10b+26)$ 4. $2(x+27)(x-11)$

5. $6c(c-1)(c-3)$ 6. $-d(d^2+2d+2)$ 7. $-(x+24)(x-18)$ 8. can not be factored

9. $5ac(b+12)(b-10)$ 10. $2y^3(y+7)(y+5)$ 11. $3x^2y(y+12)(y-6)$ 12. $-5x(x+20)(x-10)$

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