Sample Problems

Simplify each of the following expressions.

1. $\dfrac{\dfrac{1}{2}+\dfrac{1}{3}}{\dfrac{1}{2}-\dfrac{1}{3}}$

2. $\dfrac{5-\dfrac{1}{a}}{\dfrac{1}{a^2}-25}$

3. $\dfrac{1-\dfrac{1}{x^2}}{1+\dfrac{1}{x}}$

4. $\dfrac{\dfrac{4}{a^2}-1}{1-\dfrac{2}{a}}$

5. $\dfrac{\dfrac{3}{x-1}-1}{\dfrac{2}{x-1}+1}$

6. $\dfrac{2-\dfrac{3}{x+1}}{3-\dfrac{2x}{x+1}}$

7. $1-\dfrac{1}{1-\dfrac{1}{x-3}}$

8. $\dfrac{a^{-1}-b^{-1}}{a^{-2}-b^{-2}}$

9. $\dfrac{x^2-2y^{-3}}{x^{-1}+3y^2}$

10. $\left(x-y^{-2}\right)^{-3}$

Practice Problems

Simplify each of the following expressions.

1. $\dfrac{2a-\dfrac{1}{8a}}{4+\dfrac{1}{a}}$

2. $\dfrac{1}{2-\dfrac{1}{m-4}}$

3. $\dfrac{1}{1-\dfrac{1}{x-1}}$

4. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{1}{a}-\dfrac{1}{b}}$

5. $\dfrac{5-\dfrac{x}{x-1}}{2-\dfrac{x}{x+1}}$

6. $\dfrac{\dfrac{1}{x+a}+\dfrac{1}{x-a}}{\dfrac{1}{x+a}-\dfrac{1}{x-a}}$

7. $1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{q-1}}}$

8. $\dfrac{6+\dfrac{2}{x}}{\dfrac{3x+1}{4}}$

9. $\dfrac{2}{\dfrac{1}{a}+\dfrac{1}{b}}$

10. $\dfrac{1-\dfrac{y}{y-1}}{\dfrac{y}{y+1}-1}$

11. $\dfrac{\dfrac{4}{z-6}-\dfrac{7}{z}}{\dfrac{5}{z}+\dfrac{7}{z-6}}$

12. $\dfrac{\dfrac{4}{x^2}-\dfrac{3}{x}}{\dfrac{1}{x^2}+\dfrac{2}{3x}}$

13. $\dfrac{\dfrac{5}{x-3}-\dfrac{1}{x}}{\dfrac{2}{x}+\dfrac{3}{x-3}}$

14. $\dfrac{1-\dfrac{2}{3-\dfrac{4}{p}}}{1+\dfrac{2}{3-\dfrac{4}{p}}}$

15. $\dfrac{9a^{-2}-b^{-2}}{3a^{-1}+b^{-1}}$

16. $\dfrac{x^{-3}y-2x^2y^{-1}}{x^{-1}+y^{-1}}$

17. $\left(3-a^{-2}\right)^{-2}$

Sample Problems – Answers

1.) 5 2.) $-\dfrac{a}{5a+1}$ 3.) $\dfrac{x-1}{x}$ 4.) $-\dfrac{a+2}{a}$ 5.) $\dfrac{-x+4}{x+1}$ 6.) $\dfrac{2x-1}{x+3}$ 7.) $-\dfrac{1}{x-4}$

8.) $\dfrac{ab}{a+b}$ 9.) $\dfrac{x^3y^3-2x}{y^3+3xy^5}$ or $\dfrac{x\left(x^2y^3-2\right)}{y^3\left(1+3xy^2\right)}$ 10.) $\dfrac{y^6}{\left(xy^2-1\right)^3}$

Practice Problems – Answers

1) $\dfrac{4a-1}{8}$ 2) $\dfrac{m-4}{2m-9}$ 3) $\dfrac{x-1}{x-2}$ 4) $\dfrac{a+b}{b-a}$ 5) $\dfrac{4x^2-x-5}{x^2+x-2}$ 6) $-\dfrac{x}{a}$

7) $q-1$ 8) $\dfrac{8}{x}$ 9) $\dfrac{2ab}{a+b}$ 10) $\dfrac{y+1}{y-1}$ 11) $\dfrac{14-z}{4z-10}$ or $\dfrac{14-z}{2\left(2z-5\right)}$

12) $\dfrac{12-9x}{2x+3}$ or $\dfrac{-3\left(3x-4\right)}{2x+3}$ 13) $\dfrac{4x+3}{5x-6}$ 14) $\dfrac{p-4}{5p-4}$ 15) $\dfrac{3b-a}{ab}$

16.) $\dfrac{y^2-2x^5}{x^3+x^2y}$ or $\dfrac{y^2-2x^5}{x^2\left(x+y\right)}$ 17) $\dfrac{a^4}{\left(1-3a^2\right)^2}$

Sample Problems – Solutions

Simplify each of the following expressions.

1. $\dfrac{\dfrac{1}{2}+\dfrac{1}{3}}{\dfrac{1}{2}-\dfrac{1}{3}}$

Solution: We will first perform the addition indicated in the numerator and denominator. After the addition and subtraction, we will divide by multiplying by the reciprocal.

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{2}-\frac{1}{3}}=\frac{\frac{1\cdot 3}{2\cdot 3}+\frac{1\cdot 2}{3\cdot 2}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}=\frac{\frac{3}{6}+\frac{2}{6}}{\frac{3}{6}-\frac{2}{6}}=\frac{\frac{2+3}{6}}{\frac{3-2}{6}}=\frac{\frac{5}{6}}{\frac{1}{6}}=\frac{5}{\cancel{6}}\cdot\frac{\cancel{6}}{1}=\frac{5}{1}=\boxed{5}$$

2. $\dfrac{5-\dfrac{1}{a}}{\dfrac{1}{a^2}-25}$

Solution: Method 1. Start with the subtractions. We need to work with common denominators. The numerator:

$$5-\frac{1}{a}=\frac{5}{1}-\frac{1}{a}=\frac{5a}{a}-\frac{1}{a}=\frac{5a-1}{a}$$

The denominator:

$$\frac{1}{a^2}-25=\frac{1}{a^2}-\frac{25}{1}=\frac{1}{a^2}-\frac{25a^2}{a^2}=\frac{1-25a^2}{a^2}$$

Notice that the numerator here factors via the difference of squares theorem.

$$\frac{1-25a^2}{a^2}=\frac{1^2-(5a)^2}{a^2}=\frac{(1-5a)(1+5a)}{a^2}$$

Now we are ready to perform the division. To divide is to multiply by the reciprocal.

$$\frac{5-\frac{1}{a}}{\frac{1}{a^2}-25}=\frac{\frac{5a-1}{a}}{\frac{(1-5a)(1+5a)}{a^2}}=\frac{5a-1}{a}\cdot\frac{a^2}{(1-5a)(1+5a)}\qquad\text{cancel out } a$$

$$=\frac{a(5a-1)}{(1-5a)(1+5a)}$$

There is one more cancellation: $1-5a$ and $5a-1$ are opposites. We re-write $5a-1$ as $-(1-5a)$. Then

$$\frac{a(5a-1)}{(1-5a)(1+5a)}=\frac{-(1-5a)(a)}{(1-5a)(1+5a)}=\boxed{\frac{-a}{1+5a}\quad\text{or}\quad -\frac{a}{5a+1}}$$

Method 2. This method is very efficient. The main idea is to clear denominators by multiplying both numerator and denominator by the same quantity. We usually do not apply this method with usual fractions but it does work there too. The first step is to multiply both numerator and denominator by a^2, the quantity that clears the denominator in the fractions in both numerator and denominator.

$$\frac{5-\dfrac{1}{a}}{\dfrac{1}{a^2}-25}=\frac{5-\dfrac{1}{a}}{\dfrac{1}{a^2}-25}\cdot\frac{a^2}{a^2}=\frac{a^2\left(5-\dfrac{1}{a}\right)}{a^2\left(\dfrac{1}{a^2}-25\right)}=\frac{5a^2-a^2\cdot\dfrac{1}{a}}{a^2\cdot\dfrac{1}{a^2}-25a^2}=\frac{5a^2-a}{1-25a^2}$$

We now factor the expressions in the numerator and denominator so that we could see if further simplification is possible. We factor out the greatest common factor from the numerator and factor the denominator via the difference of squares theorem.

$$\frac{5a^2-a}{1-25a^2}=\frac{a\,(5a-1)}{(1-5a)\,(1+5a)}$$

Notice that $5a-1$ and $1-5a$ are oppsites. We factor out -1 from the numerator, so we re-write $5a-1$ as $-(1-5a)$. Then we cancel out the common factor.

$$\frac{a\,(5a-1)}{(1-5a)\,(1+5a)}=\frac{a\,(-1)\,(1-5a)}{(1-5a)\,(1+5a)}=\frac{a\,(-1)}{1+5a}=\boxed{\frac{-a}{5a+1}}$$

3. $\dfrac{1-\dfrac{1}{x^2}}{1+\dfrac{1}{x}}$

Solution: Method 1. We bring fractions to the common denominator and add and subtract. Then we perform the division as multiplication by the reciprocal. Finally, we factor and cancel out common factors.

$$\frac{1-\dfrac{1}{x^2}}{1+\dfrac{1}{x}}=\frac{\dfrac{1\cdot x^2}{1\cdot x^2}-\dfrac{1}{x^2}}{\dfrac{1\cdot x}{1\cdot x}+\dfrac{1}{x}}=\frac{\dfrac{x^2-1}{x^2}}{\dfrac{x+1}{x}}=\frac{x^2-1}{x^2}\cdot\frac{x}{x+1}=\frac{(x+1)\,(x-1)}{x^2}\cdot\frac{x}{x+1}=\boxed{\frac{x-1}{x}}$$

Method 2. First we multiply both numerator and denominator by x^2.

$$\frac{1-\dfrac{1}{x^2}}{1+\dfrac{1}{x}}=\frac{1-\dfrac{1}{x^2}}{1+\dfrac{1}{x}}\cdot\frac{x^2}{x^2}=\frac{x^2\left(1-\dfrac{1}{x^2}\right)}{x^2\left(1+\dfrac{1}{x}\right)}=\frac{x^2\cdot 1-x^2\cdot\dfrac{1}{x^2}}{x^2\cdot 1+x^2\cdot\dfrac{1}{x}}=\frac{x^2-1}{x^2+x}$$

We now factor and cancel out common factors.

$$\frac{x^2-1}{x^2+x}=\frac{(x+1)\,(x-1)}{x\,(x+1)}=\boxed{\frac{x-1}{x}}$$

4. $\dfrac{\dfrac{4}{a^2}-1}{1-\dfrac{2}{a}}$

Solution: Method 1

$$\frac{\dfrac{4}{a^2}-1}{1-\dfrac{2}{a}} = \frac{\dfrac{4}{a^2}-\dfrac{1}{1}}{\dfrac{1}{1}-\dfrac{2}{a}} = \frac{\dfrac{4}{a^2}-\dfrac{1\cdot a^2}{1\cdot a^2}}{\dfrac{1\cdot a}{1\cdot a}-\dfrac{2}{a}} = \frac{\dfrac{4-a^2}{a^2}}{\dfrac{a-2}{a}} = \frac{4-a^2}{a^2}\cdot\frac{a}{a-2} = \frac{(2+a)(2-a)}{a}\cdot\frac{1}{a-2}$$
$$= \frac{(2+a)(-1)(a-2)}{a(a-2)} = \boxed{-\frac{a+2}{a}}$$

Method 2. We first multiply both numerator and denominator by a^2. Then we factor all expressions and simplify the fraction.

$$\frac{\dfrac{4}{a^2}-1}{1-\dfrac{2}{a}} = \frac{a^2\left(\dfrac{4}{a^2}-1\right)}{a^2\left(1-\dfrac{2}{a}\right)} = \frac{4-a^2}{a^2-2a} = \frac{(2+a)(2-a)}{a(a-2)}$$

Notice that $a-2$ and $2-a$ are opposites. We will factor out -1 from $2-a$ in the numerator.

$$\frac{(2+a)(2-a)}{a(a-2)} = \frac{(2+a)(-1)(a-2)}{a(a-2)} = \frac{(2+a)(-1)}{a} = \boxed{-\frac{a+2}{a}}$$

5. $\dfrac{\dfrac{3}{x-1}-1}{\dfrac{2}{x-1}+1}$

Solution: Method 1.

$$\frac{\dfrac{3}{x-1}-1}{\dfrac{2}{x-1}+1} = \frac{\dfrac{3}{x-1}-\dfrac{1}{1}}{\dfrac{2}{x-1}+\dfrac{1}{1}} = \frac{\dfrac{3}{x-1}-\dfrac{1\cdot(x-1)}{1\cdot(x-1)}}{\dfrac{2}{x-1}+\dfrac{1\cdot(x-1)}{1\cdot(x-1)}} = \frac{\dfrac{3}{x-1}-\dfrac{x-1}{x-1}}{\dfrac{2}{x-1}+\dfrac{x-1}{x-1}} = \frac{\dfrac{3-(x-1)}{x-1}}{\dfrac{2+(x-1)}{x-1}} = \frac{\dfrac{3-x+1}{x-1}}{\dfrac{2+x-1}{x-1}}$$
$$= \frac{\dfrac{-x+4}{x-1}}{\dfrac{x+1}{x-1}} = \frac{-x+4}{x-1}\cdot\frac{x-1}{x+1} = \boxed{\frac{-x+4}{x+1}}$$

Method 2. We multiply both nuerator and denominator by $x-1$. We need to be careful with subtractions as we are subtracting entire expressions.

$$\frac{\dfrac{3}{x-1}-1}{\dfrac{2}{x-1}+1} = \frac{(x-1)\left(\dfrac{3}{x-1}-1\right)}{(x-1)\left(\dfrac{2}{x-1}+1\right)} = \frac{3-1(x-1)}{2+1(x-1)} = \frac{3-x+1}{2+x-1} = \frac{4-x}{x+1} = \boxed{\frac{-x+4}{x+1}}$$

6. $\dfrac{2-\dfrac{3}{x+1}}{3-\dfrac{2x}{x+1}}$

Solution: Method 1. We first work out the numerator

$$2-\frac{3}{x+1} = \frac{2}{1}-\frac{3}{x+1} \qquad \text{common denominator is } x+1$$
$$= \frac{2(x+1)}{x+1}-\frac{3}{x+1} = \frac{2(x+1)-3}{x+1} = \frac{2x+2-3}{x+1} = \frac{2x-1}{x+1}$$

Now for the denominator

$$\begin{aligned}3-\frac{2x}{x+1} &= \frac{3}{1}-\frac{2x}{x+1} \qquad \text{common denominator is } x+1\\ &= \frac{3(x+1)}{x+1}-\frac{2x}{x+1}=\frac{3(x+1)-2x}{x+1}=\frac{3x+3-2x}{x+1}=\frac{x+3}{x+1}\end{aligned}$$

Now the division:

$$\begin{aligned}\frac{2-\dfrac{3}{x+1}}{3-\dfrac{2x}{x+1}} &= \frac{\dfrac{2x-1}{x+1}}{\dfrac{x+3}{x+1}} = \qquad \text{to divide is to multiply by the reciprocal}\\ &= \frac{2x-1}{x+1}\cdot\frac{x+1}{x+3}=\boxed{\frac{2x-1}{x+3}}\end{aligned}$$

Method 2.

$$\frac{2-\dfrac{3}{x+1}}{3-\dfrac{2x}{x+1}}\cdot\frac{x+1}{x+1}=\frac{\left(2-\dfrac{3}{x+1}\right)(x+1)}{\left(3-\dfrac{2x}{x+1}\right)(x+1)}=\frac{2(x+1)-\dfrac{3}{x+1}(x+1)}{3(x+1)-\dfrac{2x}{x+1}(x+1)}=\frac{2(x+1)-3}{3(x+1)-2x}=\frac{2x+2-3}{3x+3-2x}=\boxed{\frac{2x-1}{x+3}}$$

7. $1-\dfrac{1}{1-\dfrac{1}{x-3}}$

Solution:

$$\begin{aligned}1-\frac{1}{1-\dfrac{1}{x-3}} &= 1-\frac{1}{\dfrac{1}{1}-\dfrac{1}{x-3}}=1-\frac{1}{\dfrac{1\cdot(x-3)}{1\cdot(x-3)}-\dfrac{1}{x-3}}=1-\frac{1}{\dfrac{x-3}{x-3}-\dfrac{1}{x-3}}\\ &= 1-\frac{1}{\dfrac{x-3-1}{x-3}}=1-\frac{\dfrac{1}{1}}{\dfrac{x-4}{x-3}}=1-\frac{1}{1}\cdot\frac{x-3}{x-4}=1-\frac{x-3}{x-4}\\ &= \frac{x-4}{x-4}-\frac{x-3}{x-4}=\frac{(x-4)-(x-3)}{x-4}=\frac{x-4-x+3}{x-4}=\boxed{\frac{-1}{x-4}}\end{aligned}$$

8. $\dfrac{a^{-1}-b^{-1}}{a^{-2}-b^{-2}}$

Solution: This does not even look like a complex fraction, but it will, once we re-write it positive exponents. Recall that $a^{-n}=\dfrac{1}{a^n}$. So our problem can be re-written as

$$\frac{a^{-1}-b^{-1}}{a^{-2}-b^{-2}}=\frac{\dfrac{1}{a}-\dfrac{1}{b}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}$$

Method 1.

$$\frac{\dfrac{1}{a}-\dfrac{1}{b}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}=\frac{\dfrac{b-a}{ab}}{\dfrac{b^2-a^2}{a^2b^2}}=\frac{b-a}{ab}\cdot\frac{a^2b^2}{b^2-a^2}=\frac{(b-a)\,ab}{b^2-a^2}=\frac{(b-a)\,ab}{(b-a)(b+a)}=\frac{ab}{b+a}=\boxed{\frac{ab}{a+b}}$$

Method 2.

$$\frac{\dfrac{1}{a}-\dfrac{1}{b}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}=\frac{\dfrac{1}{a}-\dfrac{1}{b}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}\cdot\frac{a^2b^2}{a^2b^2}=\frac{a^2b^2\left(\dfrac{1}{a}-\dfrac{1}{b}\right)}{a^2b^2\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)}=\frac{ab^2-a^2b}{b^2-a^2}=\frac{ab\,(b-a)}{(b-a)\,(b+a)}=\boxed{\frac{ab}{a+b}}$$

9. $\dfrac{x^2-2y^{-3}}{x^{-1}+3y^2}$

Solution: Method 1.

$$\frac{x^2-2y^{-3}}{x^{-1}+3y^2}=\frac{x^2-2\dfrac{1}{y^3}}{\dfrac{1}{x}+3y^2}=\frac{\dfrac{x^2}{1}-\dfrac{2}{y^3}}{\dfrac{1}{x}+\dfrac{3y^2}{1}}=\frac{\dfrac{x^2y^3}{y^3}-\dfrac{2}{y^3}}{\dfrac{1}{x}+\dfrac{3xy^2}{x}}=\frac{\dfrac{x^2y^3-2}{y^3}}{\dfrac{1+3xy^2}{x}}=\frac{x^2y^3-2}{y^3}\cdot\frac{x}{1+3xy^2}=\boxed{\frac{x\left(x^2y^3-2\right)}{y^3\left(1+3xy^2\right)}}$$

Method 2.

$$\frac{x^2-2y^{-3}}{x^{-1}+3y^2}=\frac{x^2-2\dfrac{1}{y^3}}{\dfrac{1}{x}+3y^2}=\frac{\dfrac{x^2}{1}-\dfrac{2}{y^3}}{\dfrac{1}{x}+\dfrac{3y^2}{1}}\cdot\frac{xy^3}{xy^3}=\frac{xy^3\left(\dfrac{x^2}{1}-\dfrac{2}{y^3}\right)}{xy^3\left(\dfrac{1}{x}+\dfrac{3y^2}{1}\right)}=\frac{x^3y^3-2x}{y^3+3xy^5}=\boxed{\frac{x\left(x^2y^3-2\right)}{y^3\left(1+3xy^2\right)}}$$

10. $\left(x-y^{-2}\right)^{-3}$

Solution: Method 1.

$$\begin{aligned}\left(x-y^{-2}\right)^{-3} &= \frac{1}{\left(x-y^{-2}\right)^3}=\frac{1}{\left(x-\dfrac{1}{y^2}\right)^3}=\frac{1}{\left(\dfrac{xy^2}{y^2}-\dfrac{1}{y^2}\right)^3}=\frac{1}{\left(\dfrac{xy^2-1}{y^2}\right)^3}=\frac{1}{\dfrac{\left(xy^2-1\right)^3}{\left(y^2\right)^3}}\\ &= 1\cdot\frac{\left(y^2\right)^3}{\left(xy^2-1\right)^3}=\boxed{\frac{y^6}{\left(xy^2-1\right)^3}}\end{aligned}$$

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