As we have seen recently, solution sets are becoming more and more complicated as we learn new types of algebraic problems such as inequalities and systems of equations. Recall that a compound statement is one that was formed from two statements, using the word *and* or *or* to connect the two statements

Example 1. Label each of the given statements as true or false.

a) 51 is a prime number.

b) Every square is a rectangle.

c) 51 is a prime number and every square is a rectangle.

d) 51 is a prime number or every square is a rectangle.

Solution: a) If we add the digits of 51, the result is 6. This indicates that 51 is divisible by 3. Indeed, $51 = 3 \cdot 17$ and so 51 has at least four divisors: 1,3,17, and 51. prime numbers have exactly two divisors. Therefore, 51 is not a prime number and so the given statement is $\boxed{\text{false}}$.

b) Every square has four right angles. Having four right angles is how we define rectangles. Therefore, the given statement is $\boxed{\text{true}}$.

c) A true and a false statements are connected with the word *and*. Such a statement is true only if both statements are true. Therefore, the given statement is $\boxed{\text{false}}$.

d) A true and a false statements are connected with the word *or*. Such a statement is true if one or both statements are true. Because one of the two statements is true, the given statement is $\boxed{\text{true}}$.

Example 2. Consider the set $S = \{1, 4, 7, 8, 13\}$. Find all elements of S for which the given statement is true.

a) $x < 8$ and x is odd

b) $x < 8$ or x is odd

Solution: a) Consider each of the following statements. Recall that a compound statement connected with the word *and* is true if both statements are true.

i) $1 < 8$ and 1 is odd This compound statement is true because both statements in the compound statement are true.

ii) $4 < 8$ and 4 is odd This compound statement is false because the second statement is false.

iii) $7 < 8$ and 7 is odd This compound statement is true because both statements in the compound statement are true.

iv) $8 < 8$ and 8 is odd This compound statement is false because both statements in the compound statement are false.

v) $13 < 8$ and 13 is odd This compound statement is false because the first statement is false.

Therefore, only $\boxed{1 \text{ and } 7}$ are the elements of S for which the given compound statement is true.

b) Consider each of the following statements. Recall that a compound statement connected with the word *or* is true if at least one of the statements is true.

i) $1 < 8$ or 1 is odd This compound statement is true because both statements in the compound statement are true.

ii) $4 < 8$ or 4 is odd This compound statement is true because the first statement is true.

iii) $7 < 8$ or 7 is odd This compound statement is true because both statements in the compound statement are true.

iv) $8 < 8$ or 8 is odd This compound statement is false because both statements in the compound statement are false.

v) $13 < 8$ or 13 is odd This compound statement is true because the second statement is true.

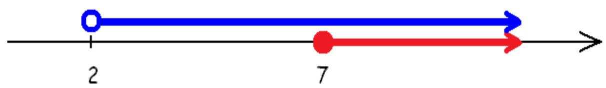
Therefore, $\boxed{1,\ 4,\ 7 \text{ and } 13}$ are the elements of S for which the given compound statement is true.

Compund inequalities will be just that: two statements of inequality, connected with the word *and* or *or*.

Example 3. Find all numbers that satisfy the given compound inequalities.

a) $x > 2$ and $x \geq 7$ b) $x > 2$ or $x \geq 7$

Solution: Before we would solve the compound inequalities, let us depict on a number line the set of all numbers for which $x > 2$ and also, the set of all numbers for which $x \geq 7$.



a) The numbers greater than 2 have the blue line above them. The numbers greater than or equal to 7 are marked with red. Because the two statements are connected with the word *and*, we are looking for numbers for which both statements are true. This means that the numbers we are looking for are marked red and have the blue line above them. That set is clearly $\boxed{[7, \infty)}$.

b) Now the two statements are connected with the word *or*. This means that we are looking for numbers for which at least one of the statements are true. This means that the numbers we are looking for are marked red or have the blue line above them, or both. That set is clearly $\boxed{(2, \infty)}$.

Example 4. Find all numbers that satisfy the given compound inequalities.

a) $x < 1$ and $x \geq 0$ b) $x < 1$ or $x \geq 0$

Solution: Before we would solve the compound inequalities, let us depict on a number line the set of all numbers for which $x < 1$ and also, the set of all numbers for which $x \geq 1$.

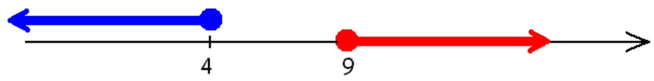
a) The numbers less than 1 are marked with red. The numbers greater than or equal to 0 have the blue line above them. Because the two statements are connected with the word *and*, we are looking for numbers for which both statements are true. This means that the numbers we are looking for are marked red and have the blue line above them. That set is clearly $\boxed{[0, 1)}$. In case of a compound inequality of this type, we can re-write $x < 1$ and $x \geq 0$ in a shorter form, as $0 \leq x < 1$.

b) Now the two statements are connected with the word *or*. This means that we are looking for numbers for which at least one of the statements are true. This means that the numbers we are looking for are marked red or have the blue line above them, or both. That set is clearly the set of all real numbers, $\boxed{\mathbb{R}}$.

Example 5. Find all numbers that satisfy the given compound inequalities.

a) $x \leq 4$ and $x \geq 9$ b) $x \leq 4$ or $x \geq 9$

Solution: Before we would solve the compound inequalities, let us depict on a number line the set of all numbers for which $x \leq 4$ and also, the set of all numbers for which $x \geq 9$.



a) The numbers less than or equal to 4 have the blue line above them. The numbers greater than or equal to 9 are marked with red. Because the two statements are connected with the word *and*, we are looking for numbers for which both statements are true. This means that the numbers we are looking for are marked red and have the blue line above them. There is no such number and so the solution set is the empty set, $\boxed{\varnothing}$.

b) Now the two statements are connected with the word *or*. This means that we are looking for numbers for which at least one of the statements are true. This means that the numbers we are looking for are marked red or have the blue line above them, or both. That set is $\boxed{(-\infty, 4] \cup [9, \infty)}$.

Practice Problems

Solve each of the given compound inequalities. Present your answer in interval form.

1. a) $5x - 1 > -16$ and $-x + 8 \geq 3$
   b) $5x - 1 > -16$ or $-x + 8 \geq 3$
2. a) $3\left((2x-5)-(x-4)\right) \leq 2x - 7$ or $(x-6) - 2(x-4) \leq -x - 10$
   b) $3\left((2x-5)-(x-4)\right) \leq 2x - 7$ and $(x-6) - 2(x-4) \leq -x - 10$
3. a) $\frac{2}{5}x - 4 > -\frac{1}{2}x + 5$ and $4(x-3) - 2(x-3) < -2x + 14$
   b) $\frac{2}{5}x - 4 > -\frac{1}{2}x + 5$ or $4(x-3) - 2(x-3) < -2x + 14$
4. a) $\frac{3}{4}x - 2 > x - 3$ or $2x + 3 > 5x - 21$
   b) $\frac{3}{4}x - 2 > x - 3$ and $2x + 3 > 5x - 21$
5. a) $(x+3)^2 - (x-1)^2 \leq -5(x+1)$ and $\frac{2}{3}x - \frac{3}{5} > \frac{1}{5}x - \frac{2}{15}$
   b) $(x+3)^2 - (x-1)^2 \leq -5(x+1)$ or $\frac{2}{3}x - \frac{3}{5} > \frac{1}{5}x - \frac{2}{15}$

Answers

1. a) $(-3, 5]$ b) $\mathbb{R} = (-\infty, \infty)$ 2. a) $(-\infty, 0]$ b) $(-\infty, -4]$ 3. a) $\varnothing$ b) $(-\infty, 5) \cup (10, \infty)$

4. a) $(-\infty, 8)$ b) $(-\infty, 4)$ 5. a) $\varnothing$ b) $(-\infty, -1] \cup (1, \infty)$

