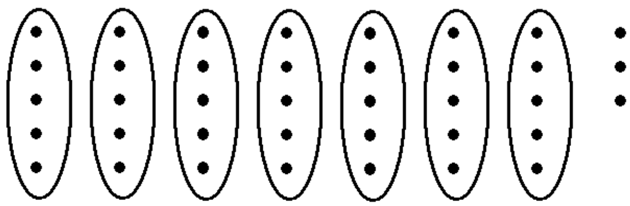
Theorem (Division with Remainder) For every integers N and m, $m \neq 0$, there exist unique integers q and r such that

$$N = mq + r \quad \text{and} \quad r < m$$

For example, if $N = 38$ and $m = 5$, then the quotient is $q = 7$ and the remainder is $r = 3$. The picture below illustrates the division $38 \div 5 = 7$ R 3, or, in other form: $38 = 5 \cdot 7 + 3$.



Note that there are two different ways to represent this division: $38 \div 5 = 7$ R 3 and $\dfrac{38}{5} = 7\dfrac{3}{5}$
This theorem, also called the Euclidean algorithm, is very fundamental to mathematics.

Example 1. Perform the given division with remainder. $6071 \div 17$
Solution: One way would be is to perform long division. Another is to utilize our calculator. First we enter the division into the calculator and see the result as a decimal: $357.117\,647\,058\,8$. Note that this is NOT division with remainder. However, we can quickly adjust this. Since the result is between 357 and 358, we already know that the quotient is 357 and now we need to only find the remainder.

$$6071 \div 17 = 319 \text{ R } ?$$

If the quotient is 357, then we have already accounted for $17 \cdot 357 = 6069$. How much is missing from 6071? Clearly 2. Thus the answer is: $6071 \div 17 = 319$ R 2.

How can we check? We should look for two things: did we account for all 6071? Indeed, $17 \cdot 357 + 2 = 6071$. The other question is: Is the remainder less than the divisor? Indeed, $2 < 17$ and so our solution is correct.

Example 2. Perform the given division with remainder. $8271 \div 45 = 183.8$
Solution: First we enter the division into the calculator and see the result as a decimal: 183.8. Note that this is NOT division with remainder. However, we can quickly adjust this. Since the result is between 183 and 184, we already know that the quotient is 183 and now we need to only find the remainder. If the quotient is 183, then we have already accounted for $183 \cdot 45 = 8235$. How much is missing from 6071? $8271 - 8235 = 36$ Thus the answer is: $8271 \div 45 = 184$ R 36.
We check: $183 \cdot 45 + 36 = 8271$, and $36 < 45$ and so the remainder is less than the divisor. So our solution is correct.

An Application: Orbits

Example 3. What is the last digit of 2^{99}?
Solution: Let us first investigate what the last digits of smaller 2−powers are.

$2^1 = 2$	$2^5 = 3\boxed{2}$	$2^9 = 51\boxed{2}$
$2^2 = 4$	$2^6 = 6\boxed{4}$	$2^{10} = 102\boxed{4}$
$2^3 = 8$	$2^7 = 12\boxed{8}$	$2^{11} = 204\boxed{8}$
$2^4 = 1\boxed{6}$	$2^8 = 25\boxed{6}$	$2^{12} = 409\boxed{6}$

Before the two-powers become uncomfortably large, we can observe a repeating pattern: the last digits being

$$2, 4, 8, 6, 2, 4, 8, 6, 2, 4, 8, 6, \ldots.$$

There are only 4 numbers appearing as last digit, following each other in a repeating pattern. The question only remains: where does 99 land in this pattern?

One way to determine that is to apply division with remainder. Consider the last row:

$$2^4 = 1\boxed{6}, \quad 2^8 = 25\boxed{6}, \quad 2^{12} = 409\boxed{6},$$

We can see that if the exponent is divisible by 4, then the last digit is 6. Consider now the first row:

$$2^1 = \boxed{2}, \quad 2^5 = 3\boxed{2}, \quad 2^9 = 51\boxed{2},$$

These exponents, $1, 5, 9, 13, \ldots$ are right after a number divisible by 4. These are numbers that result in a remainder 1 when divided by 4 and can be expressed as $4k + 1$ where k is an integer. In the next row,

$$2^2 = \boxed{4}, \quad 2^6 = 6\boxed{4}, \quad 2^{10} = 102\boxed{4},$$

the exponents are even but not divisible by 4. These can also be expressed as numbers that result in a remainder 2 when divided by 4 and can be expressed as $4k + 2$ where k is an integer. In the last row,

$$2^3 = \boxed{8}, \quad 2^7 = 12\boxed{8}, \quad 2^{11} = 204\boxed{8},$$

the exponents are even but not divisible by 4. These can also be expressed as numbers that result in a remainder 2 when divided by 4 and can be expressed as $4k + 2$ where k is an integer. We amend our result with these labels:

$2^1 = 2$	$2^5 = 3\boxed{2}$	$2^9 = 51\boxed{2}$	exponent: $4k + 1$	last digit: 2
$2^2 = 4$	$2^6 = 6\boxed{4}$	$2^{10} = 102\boxed{4}$	exponent: $4k + 2$	last digit: 4
$2^3 = 8$	$2^7 = 12\boxed{8}$	$2^{11} = 204\boxed{8}$	exponent: $4k + 3$	last digit: 8
$2^4 = 1\boxed{6}$	$2^8 = 25\boxed{6}$	$2^{12} = 409\boxed{6}$	exponent: $4k$	last digit: 6

We can find where 99 lands in this pattern if we divide it by 4 and focus on the remainder.

$$99 \div 4 = 24 \text{ R } 3$$

Since the remainder is 3, 99 lands in the second to last row with exponents $3, 7, 11, 15, \ldots = 4k + 3$, and so the last digit of 2^{99} is 8.

Practice Problems

1. Perform the indicated divisions with remainders.

   a) $132 \div 7$ b) $1145 \div 12$ c) $918 \div 8$ d) $201 \div 12$

2. Find the last digit of each of the following numbers.

   a) 7^{2017} b) 9^{120} c) 2^{1001} d) 3^{286}

Answers - Practice Problems

1. a) 18 R 6 b) 95 R 5 c) 114 R 6 d) 16 R 9
2. a) 7 b) 1 c) 2 d) 9

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