In what follows, we will study rational equations. As we will see, rational equations are quite interesting. They often require nonequivalent steps. Recall first a few definitions.

> **Definition:** When solving equations, an **equivalent step** is one that preserves the solution set from one line to the next. If a step is not like that, we call it a **nonequivalent step**.

Most steps we have been using in solving equations are equivalent steps. However, solving rational equations will sometimes require nonequivalent steps.

> **Definition:** A **rational expression** is a ratio of two polynomials.

For example, $\dfrac{x-2}{x+7}$, $\dfrac{x+8}{x^2+4}$, and $\dfrac{a^2-2a+1}{a^2-9}$ are all rational expressions. Rational expressions are slightly different from the algebraic expressions we have seen so far. If we consider algebraic expressions such as x^2, $|2x-3|+1$ and x^3-x^2+5x-7 can be evaluated for every value of x. Every real number can be squared. On the other hand, $\dfrac{x-2}{x+7}$ will not work well with $x=-7$. This is because of division by zero: if $x=-7$, then

$$\frac{x-2}{x+7}=\frac{-7-2}{-7+7}=\frac{-9}{0}=\text{undefined}$$

> **Definition:** The **domain of an expression** is the set of all real numbers that can be meaningfully substituted into the expression.

This is to say that the domain of an expression is the set of real numbers for which we get a number and not undefined. For example, the domain of $\dfrac{x-2}{x+7}$ is the set of all real numbers except $x=-7$. In set notation, it is $\{x\in\mathbb{R}: x\neq -7\}$. Or the domain of the expression $\sqrt{x}$ is the set of all non-negative real numbers, $\{x\in\mathbb{R}: x\geq 0\}$ or $[0,\infty)$ in interval notation. This is because the square root of a negative number is undefined.

At this point, there are only two ways we can get 'undefined' when evaluating expressions are: division by zero and a negative number under an even root such as $\sqrt{\quad}$, $\sqrt[4]{\quad}$, $\sqrt[6]{\quad}$, etc.

Example 1. Find the domain for each of the following expressions.

a) $\dfrac{x+6}{x^2-25}$ b) $\dfrac{2x}{x^2-4x-12}$ c) $\dfrac{x^2-1}{x^2+1}$

Solution: a) The only way this expression can be undefined is by division by zero. We know this because there are no radicals in the expression. Every number can be squared or added 6 to it. The only way this expression can be undefined because of divsion by zero. Therefore, the domain of $\dfrac{x+6}{x^2-25}$ is the set of all real numbers, for which the denominator is not zero. We solve the quadratic eqation $x^2-25=0$ and get that $x=-5$ and 5 are the solutions. These are the only two numbers for which we get undefined. So the domain of this expression is $\boxed{\{x\in\mathbb{R}: x\neq\pm 5\}}$.

b) Since there are no radicals in this expression, the only worry is division by zero. The misbehaving numbers will only be those for which the denominator is zero. So we solve the equation $x^2-4x-12=0$.

$$\begin{aligned} x^2-4x-12 &= 0 \\ (x+2)(x-6) &= 0 \quad \Longrightarrow \quad x_1=-2, x_2=6 \end{aligned}$$

Therefore, the domain of ow can the absolute value of a number be 0? There is only one way: if the number is 0. Therefore, the domain of $\dfrac{2x}{x^2-4x-12}$ is the set of all real numbers except for -2 and 6. Using set notation, this is $\boxed{\{x\in\mathbb{R} : x\neq -2,6\}}$. The same set can be also expressed using interval notation: $(-\infty,-2)\cup(2,6)\cup(6,\infty)$.

c) Since there are no radicals in this expression, the only worry is division by zero. The only problematic numbers will only be those for which the denominator is zero. So we solve the equation $x^2+1=0$. The expression x^2+1 cannot be factored over the real numbers. The equation $x^2=-1$ has no real solution. Therefore, there is no danger of division by zero. We will always get a real number, no matter what number we substitute for x. So the domain of $\dfrac{x^2-1}{x^2+1}$ is the set of all real numbers, $\boxed{\mathbb{R}}$. The same set can be also expressed in interval notation as $(-\infty,\infty)$.

When solving rational equations, the domain of the expressions plays an interesting and sometimes significant role.

Example 2. Solve the given rational equation. $\dfrac{3x-1}{x-8}=\dfrac{2x+7}{x-8}$

Solution: Both sides of the equation are rational expressions with the same denominator. We could clear the denominator by mutiplying both sides by $x-8$. This way the equation will become a simple linear one.

$$\begin{aligned} \frac{3x-1}{x-8} &= \frac{2x+7}{x-8} && \text{multiply by } x-8 \\ 3x-1 &= 2x+7 && \text{subtract } 2x \\ x-1 &= 7 && \text{add } 1 \\ x &= 8 \end{aligned}$$

We check: if $x=8$, then LHS $=\dfrac{3\cdot 8-1}{8-8}=\dfrac{23}{0}=$ undefined. We do not even need to consider the other side. Since 8 makes one side undefined, it cannot be a solution of this equation. Therefore, this equation has $\boxed{\text{no solution}}$.

What happened here? Why did we get a solution that proved to be not a solution? Such a solution is called an **extreneous solution**. The number 8 is a solution of the second line, $3x-1=2x+7$, but not of the first line, $\dfrac{3x-1}{x-8}=\dfrac{2x+7}{x-8}$. Therefore, multiplying by $x-8$ proved to be a non-equivalent step. This is because multiplying by $x-8$ changed the domain of the expressions making up the equation. 8 is in the domain of the second line, but not of the first line.

There are several lessons from this: first, that we HAVE to check the solutions of a rational equation. In case of equivalent steps, checking is a matter of making sure we didn't make a mistake. In this case, we could have an extreneous solution even if we did everything correctly.

Example 3. Solve the given rational equation. $x-\dfrac{6x}{x+3}=\dfrac{x+21}{x+3}$

Solution: In order to clear the denominator, we will multiply both sides by $x+3$. Because this is a nonequivalent step, we must check our solution.

$$\begin{aligned}
x-\frac{6x}{x+3} &= \frac{x+21}{x+3} && \text{multiply by } x+3 \\
(x+3)\left(x-\frac{6x}{x+3}\right) &= (x+3)\left(\frac{x+21}{x+3}\right) \\
(x+3)x-\cancel{(x+3)}\cdot\frac{6x}{\cancel{x+3}} &= \cancel{(x+3)}\cdot\left(\frac{x+21}{\cancel{x+3}}\right) \\
x(x+3)-6x &= x+21 \\
x^2+3x-6x &= x+21 && \text{combine like terms} \\
x^2-3x &= x+21 && \text{subtract } x \\
x^2-4x &= 21 && \text{subtract } 21 \\
x^2-4x-21 &= 0 && \text{factor} \\
(x-7)(x+3) &= 0 \quad \Longrightarrow \quad x_1=7 \quad x_2=-3
\end{aligned}$$

We check: if $x=7$, then

$$\begin{aligned}
\text{LHS} &= 7-\frac{6\cdot 7}{7+3}=7-\frac{42}{10}=7-\frac{21}{5}=\frac{35}{5}-\frac{21}{5}=\frac{14}{5} \quad \text{and} \\
\text{RHS} &= \frac{7+21}{7+3}=\frac{28}{10}=\frac{14}{5} \quad \checkmark
\end{aligned}$$

and so 7 is a solution of the equation.

If $x=-3$, then

$$\text{LHS}=-3-\frac{6(-3)}{-3+3}=-3-\frac{-18}{0}=\text{undefined}$$

Therefore, $x=-3$ is not a solution of the equation. The only solution is $\boxed{7}$.

Naturally, not all rational equations have extrenous solutions. It is also worth noting that after we cleared the denominator, we can end up with either a linear or a quadratic equation.

Sample Problems

Solve each of the given equations.

1. $\dfrac{3x}{2x-1}=6$

2. $\dfrac{5}{x+7}-\dfrac{1}{6}=\dfrac{8}{4x+28}$

3. $\dfrac{x}{x+1}+\dfrac{4}{x+7}=\dfrac{5x-1}{x^2+8x+7}$

4. $\dfrac{1}{4y}-\dfrac{1}{2}=\dfrac{1}{y}$

5. $\dfrac{3}{x-5}-\dfrac{1}{x+3}=\dfrac{10}{x^2-2x-15}$

6. $\dfrac{x}{x-7}-\dfrac{3}{x+3}=\dfrac{4x+42}{x^2-4x-21}$

Practice Problems

Solve each of the given equations.

1. $\dfrac{3x}{7x+6}=1$
2. $\dfrac{6}{3x-1}=2$
3. $\dfrac{3}{x-2}-\dfrac{8}{x+2}=\dfrac{1}{6}$
4. $\dfrac{x}{x-3}+\dfrac{2}{x+9}=\dfrac{19x-21}{x^2+6x-27}$
5. $\dfrac{x}{x+6}+\dfrac{5}{x+2}=\dfrac{-8x-24}{x^2+8x+12}$
6. $\dfrac{12}{x-5}-\dfrac{6}{x-2}=3$
7. $\dfrac{x}{x+3}-\dfrac{1}{x-7}=\dfrac{-4x+18}{x^2-4x-21}$
8. $\dfrac{1}{x-6}-1=\dfrac{5}{x-6}$
9. $\dfrac{3}{x-7}+\dfrac{x+7}{x}=\dfrac{7x-28}{x^2-7x}$
10. $\dfrac{10}{x}-\dfrac{4}{x-1}=1$
11. $\dfrac{x}{x-2}=\dfrac{x-2}{x}$
12. $\dfrac{12}{x-1}+\dfrac{14}{x}=4$
13. $\dfrac{x}{x+1}+\dfrac{7}{x-3}=\dfrac{-x+3}{x^2-2x-3}$
14. $\dfrac{2x-1}{x+3}-\dfrac{1}{x-7}=2+\dfrac{38-6x}{x^2-4x-21}$

Answers

1. $-\frac{3}{2}$ 2. $\frac{4}{3}$ 3. $-34, 4$ 4. 5 (3 is extreneous) 5. -9 (-6 is extreneous)

6. $1, 8$ 7. no solution (both $-3, 7$ are extreneous) 8. 2 9. -3 (7 is extreneous)

10. $2, 5$ 11. 1 12. $\frac{1}{2}, 7$ 13. -4 (-1 is extreneous) 14. 4

Sample Problems Solutions

Solve each of the given equations.

1. $\dfrac{3x}{2x-1} = 6$

   Solution: We will clear the denominator by multiplying both sides by $2x-1$.

$$\begin{aligned}
\frac{3x}{2x-1} &= 6 && \text{multipliy by } 2x-1 \\
3x &= 6(2x-1) && \text{distribute} \\
3x &= 12x-6 && \text{subtract } 3x \\
0 &= 9x-6 && \text{add } 6 \\
6 &= 9x && \text{divide by } 9 \\
\frac{6}{9} &= x \quad \Longrightarrow \quad x = \frac{2}{3}
\end{aligned}$$

We check: if $x = \dfrac{2}{3}$, then

$$\text{LHS} = \frac{3\left(\frac{2}{3}\right)}{2\left(\frac{2}{3}\right)-1} = \frac{2}{\frac{4}{3}-1} = \frac{2}{\frac{4}{3}-\frac{3}{3}} = \frac{2}{\frac{1}{3}} = 2\left(\frac{3}{1}\right) = 6 = \text{RHS} \checkmark$$

, and so our solution, $\boxed{x = \dfrac{2}{3}}$ is correct.

2. $\dfrac{5}{x+7} - \dfrac{1}{6} = \dfrac{8}{4x+28}$

   Solution: It is important to notice that we do not need to multiply by $(x+7)(4x+28)$, because $4x+28 = 4(x+7)$. This is important because if we multiply by more than what is needed, we might end up with a cubic equation that we cannot solve.

$$\begin{aligned}
\frac{5}{x+7} - \frac{1}{6} &= \frac{8}{4x+28} && \text{factor } 4x+28 \\
\frac{5}{x+7} - \frac{1}{6} &= \frac{8}{4(x+7)} && \text{multiply by } 12(x+7) \\
12(x+7)\left(\frac{5}{x+7} - \frac{1}{6}\right) &= 12(x+7)\left(\frac{8}{4(x+7)}\right) \\
12(x+7)\cdot\frac{5}{x+7} - 12(x+7)\cdot\frac{1}{6} &= 12(x+7)\left(\frac{8}{4(x+7)}\right) && \text{cancel out terms} \\
12\cancel{(x+7)}\cdot\frac{5}{\cancel{x+7}} - 12(x+7)\cdot\frac{1}{6} &= \cancel{4}\cdot 3\cancel{(x+7)}\left(\frac{8}{\cancel{4}\cancel{(x+7)}}\right) \\
60 - 2(x+7) &= 24 \\
60 - 2x - 14 &= 24 \\
-2x + 46 &= 24 \\
-2x &= -22 \\
x &= 11
\end{aligned}$$

We check: if $x = 11$, then

$$\begin{aligned}
\text{LHS} &= \frac{5}{11+7} - \frac{1}{6} = \frac{5}{18} - \frac{1}{6} = \frac{5}{18} - \frac{3}{18} = \frac{2}{18} = \frac{1}{9} \\
\text{RHS} &= \frac{8}{4(11)+28} = \frac{8}{44+28} = \frac{8}{72} = \frac{1}{9} \checkmark
\end{aligned}$$

Thus our solution, $\boxed{x = 11}$ is correct.

3. $\dfrac{x}{x+1} + \dfrac{4}{x+7} = \dfrac{5x-1}{x^2+8x+7}$

Solution: It is important to notice that we do not need to multiply by $(x+7)(x+1)\left(x^2+8x+7\right)$, because $x^2+8x+7 = (x+7)(x+1)$. This is important because if we multiply by more than what is needed, we might end up with a 4th or 5th degree equation that we cannot solve.

$$\begin{aligned}
\frac{x}{x+1} + \frac{4}{x+7} &= \frac{5x-1}{x^2+8x+7} \\
\frac{x}{x+1} + \frac{4}{x+7} &= \frac{5x-1}{(x+7)(x+1)} && \text{multiply by } (x+7)(x+1) \\
(x+7)(x+1)\left(\frac{x}{x+1} + \frac{4}{x+7}\right) &= (x+7)(x+1)\frac{5x-1}{(x+7)(x+1)} \\
(x+7)(x+1)\frac{x}{x+1} + (x+7)(x+1)\frac{4}{x+7} &= (x+7)(x+1)\frac{5x-1}{(x+7)(x+1)} && \text{cancel out terms} \\
(x+7)\cancel{(x+1)}\frac{x}{\cancel{x+1}} + \cancel{(x+7)}(x+1)\frac{4}{\cancel{x+7}} &= \cancel{(x+7)}\cancel{(x+1)}\frac{5x-1}{\cancel{(x+7)}\cancel{(x+1)}} \\
x(x+7) + 4(x+1) &= 5x-1 \\
x^2+7x+4x+4 &= 5x-1 \\
x^2+11x+4 &= 5x-1 \\
x^2+6x+4 &= -1 \\
x^2+6x+5 &= 0 \\
(x+5)(x+1) &= 0 \implies x_1 = -5 \quad x_2 = -1
\end{aligned}$$

We check: if $x = -5$, then

$$\begin{aligned}
\text{LHS} &= \frac{-5}{-5+1} + \frac{4}{-5+7} = \frac{-5}{-4} - \frac{4}{-2} = \frac{5}{4} + 2 = \frac{5}{4} + \frac{8}{4} = \frac{13}{4} \\
\text{RHS} &= \frac{5(-5)-1}{(-5)^2+8(-5)+7} = \frac{-26}{25-40+7} = \frac{-26}{-8} = \frac{13}{4} \checkmark
\end{aligned}$$

Thus $x = -5$ works. And if $x = -1$, then

$$\text{LHS} = \frac{-1}{-1+1} + \frac{4}{-1+7} = \frac{-1}{0} + \frac{4}{6} = \text{undefined}$$

Since $x = -1$ is not even in the domain of the first line, it cannot be a solution. So our only solution is $\boxed{x = -5}$.

4. $\dfrac{1}{4y} - \dfrac{1}{2} = \dfrac{1}{y}$

Solution:

$$\begin{aligned}
\frac{1}{4y} - \frac{1}{2} &= \frac{1}{y} && \text{multiply by } 4y \\
4y\left(\frac{1}{4y} - \frac{1}{2}\right) &= 4y \cdot \frac{1}{y} \\
4y \cdot \frac{1}{4y} - 4y \cdot \frac{1}{2} &= 4 \\
1 - 2y &= 4 && \text{subtract } 1 \\
-2y &= 3 && \text{divide by } -2 \\
y &= -\frac{3}{2}
\end{aligned}$$

We check: if $y = -\dfrac{3}{2}$, then LHS $= \dfrac{1}{4\left(-\dfrac{3}{2}\right)} - \dfrac{1}{2} = \dfrac{1}{-6} - \dfrac{1}{2} = -\dfrac{1}{6} - \dfrac{3}{6} = -\dfrac{4}{6} = -\dfrac{2}{3}$ and

RHS $= \dfrac{1}{\left(-\dfrac{3}{2}\right)} = -\dfrac{2}{3}$ ✓

Thus our solution, $\boxed{y = -\dfrac{3}{2}}$ is correct.

5. $\dfrac{3}{x-5} - \dfrac{1}{x+3} = \dfrac{10}{x^2 - 2x - 15}$

Solution: Let us first factor the quadratic denominator. $x^2 - 2x - 15 = (x+3)(x-5)$. Therefore, we only need to multiply both sides by $(x+3)(x-5)$.

$$\begin{aligned}
\frac{3}{x-5} - \frac{1}{x+3} &= \frac{10}{x^2 - 2x - 15} \\
\frac{3}{x-5} - \frac{1}{x+3} &= \frac{10}{(x+3)(x-5)} && \text{multiply by } (x+3)(x-5) \\
(x+3)(x-5)\left(\frac{3}{x-5} - \frac{1}{x+3}\right) &= (x+3)(x-5)\frac{10}{(x+3)(x-5)} \\
(x+3)(x-5)\frac{3}{x-5} - (x+3)(x-5)\frac{1}{x+3} &= (x+3)(x-5)\frac{10}{(x+3)(x-5)} \\
(x+3)\cancel{(x-5)}\frac{3}{\cancel{x-5}} - \cancel{(x+3)}(x-5)\frac{1}{\cancel{x+3}} &= \cancel{(x-5)}\cancel{(x+3)}\frac{10}{\cancel{(x+3)}\cancel{(x-5)}} && \text{cancel terms} \\
3(x+3) - (x-5) &= 10 \\
3x + 9 - x + 5 &= 10 \\
2x + 14 &= 10 \\
2x &= -4 \\
x &= -2
\end{aligned}$$

We check: if $x=-2$, then

$$\begin{aligned} \text{LHS} &= \frac{3}{-2-5}-\frac{1}{-2+3}=\frac{3}{-7}-\frac{1}{1}=-\frac{3}{7}-\frac{7}{7}=-\frac{10}{7} \\ \text{RHS} &= \frac{10}{(-2)^2-2(-2)-15}=\frac{10}{4+4-15}=\frac{10}{-7}=-\frac{10}{7}\ \checkmark \end{aligned}$$

Thus our solution, $\boxed{x=-2}$ is correct.

6. $\dfrac{x}{x-7}-\dfrac{3}{x+3}=\dfrac{4x+42}{x^2-4x-21}$

Solution:

$$\begin{aligned} \frac{x}{x-7}-\frac{3}{x+3} &= \frac{4x+42}{x^2-4x-21} \\ \frac{x}{x-7}-\frac{3}{x+3} &= \frac{4x+42}{(x-7)(x+3)} && \text{multiply by } (x-7)(x+3) \\ (x-7)(x+3)\left(\frac{x}{x-7}-\frac{3}{x+3}\right) &= (x-7)(x+3)\frac{4x+42}{(x-7)(x+3)} \\ (x-7)(x+3)\frac{x}{x-7}-(x-7)(x+3)\frac{3}{x+3} &= (x-7)(x+3)\frac{4x+42}{(x-7)(x+3)} && \text{cancel out terms} \\ \cancel{(x-7)}(x+3)\frac{x}{\cancel{x-7}}-(x-7)\cancel{(x+3)}\frac{3}{\cancel{x+3}} &= \cancel{(x-7)}\cancel{(x+3)}\frac{4x+42}{\cancel{(x-7)}\cancel{(x+3)}} \\ x(x+3)-3(x-7) &= 4x+42 \\ x^2+3x-3x+21 &= 4x+42 \\ x^2-4x-21 &= 0 \\ (x+3)(x-7) &= 0 \quad \Longrightarrow x_1=-3 \quad x_2=7 \end{aligned}$$

We check: if $x=7$, then LHS $=\dfrac{7}{7-7}-\dfrac{3}{7+3}=\dfrac{7}{0}-\dfrac{3}{10}=$ undefined

Thus $x=7$ does not work, it is an extreneous solution. And if $x=-3$, then

$$\text{LHS}=\frac{-3}{-3-7}+\frac{3}{-3+3}=\frac{-3}{-10}+\frac{3}{0}=\text{undefined}$$

Since $x=-3$ is not even in the domain of the first line, it cannot be a solution. So it is also an extreneous solution. So this equation has $\boxed{\text{no solution}}$.

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