Sample Problems

1. Completely factor each of the following.

   a) $3a^3 - 27ab^2$ c) $2p^4 - 162$ e) $357ab^2 - 30ab^2x - 3ab^2x^2$

   b) $100x - x^2 - 2419$ d) $x^2 - 4x + 7$ f) $20x + 5x^3$

2. Solve each of the following equations. Make sure to check your solution.

   a) $8x^3 = 50x^2$ b) $8p^3 = 50p$ c) $2x^3 = 20x^2 + 1750x$

3. Word Problems

   a) One number is 18 less than the other. Find these numbers if their product is 1600.

   b) We throw an object upward from the top of a 1200 ft tall building. The height of the object, (measured in feet) t seconds after we threw it is

   $$h(t) = -16t^2 + 160t + 1200$$

   i) Where is the object 3 seconds after we threw it?

   ii) How long does it take for the object to hit the ground?

   c) Find all numbers that satisfy the following condition: if we square the number, we get back the same number.

   d) Find all numbers that satisfy the following condition: if we raise the number to the third power, the result is four times the original number.

   e) The area of a rectangle is 1260 m^2. Find the dimensions of the rectangle if we know that one side is 48 m longer than three times the other side.

Practice Problems

1. Completely factor each of the following.

   a) $a^4 - 16$ d) $36x^2y^3 + 4x^4y^3$ g) $x^2 + 16x + 73 =$

   b) $3p^2 - 12p - 288$ e) $-2x^4 + 162$

   c) $600ab^2 - 6ab^4$ f) $5a^3b^2 - 15ab$

2. Solve each of the following equations. Make sure to check your solutions.

   a) $3x^3 = 75x$ b) $x^3 - 270x = 3x^2$ c) $(x+1)(1-2x) = -3x^2 + 7x + 34$

3. Word Problems.

   a) One number is 32 less than the other. Find these numbers if their product is -135.

   b) The difference between two numbers is 7, their product is 228. Find these numbers.

   c) We throw an object upward from the top of a 112 ft tall building. The height of the object, (measured in feet) t seconds after we threw it is

   $$h(t) = -16t^2 + 96t + 112$$

   i) Where is the object 3 seconds after we threw it?

   ii) How long does it take for the object to hit the ground?

   d) The area of a rectangle is 1100 m^2. Find the dimensions of the rectangle if we know that one side is 60 m shorter than five times another side.

Sample Problems - Answers

1. a) $3a(a+3b)(a-3b)$ b) $-(x-41)(x-59)$ c) $2(p^2+9)(p+3)(p-3)$
   d) does not factor over the real numbers e) $-3ab^2(x+17)(x-7)$ f) $5x(x^2+4)$
2. a) 0 and $\frac{25}{4}$ b) $-\frac{5}{2}$, 0, and $\frac{5}{2}$ c) $35, 0, -25$
3. Word Problems
   a) $32, 50$ and $-50, -32$ b) i) 1536 ft ii) 15 seconds c) $0, 1$
   d) $0, 2, -2$ e) 14 m by 90 m

Practice Problems - Answers

1. a) $(a-2)(a+2)(a^2+4)$ b) $3(p+8)(p-12)$ c) $-6ab^2(b-10)(b+10)$
   d) $4x^2y^3(x^2+9)$ e) $-2(x^2+9)(x+3)(x-3)$ f) $5ab(a^2b-3)$
   g) does not factor over the real numbers
2. a) $-5, 0, 5$ b) $18, 0, -15$ c) $-3, 11$
3. a) -5 with 27 and -27 with 5 b) $12, 19$ and $-19, -12$ c) i) 256 ft ii) 7 seconds
   d) 22 m by 50 m

Sample Problems - Solutions

1. Completely factor each of the following.

a) $3a^3 - 27ab^2 =$ $3a(a+3b)(a-3b)$

Solution: We start with the greatest common factor (or GCF).

$$\begin{aligned} 3a^3 - 27ab^2 &= \quad \text{factor out GCF} \\ 3a\left(a^2 - 9b^2\right) &= \quad \text{re-write } 9b^2 \text{ as } (3b)^2 \\ 3a\left(a^2 - (3b)^2\right) &= \quad \text{factor via the difference of squares theorem} \\ &= 3a(a+3b)(a-3b) \end{aligned}$$

We check by multiplication:

$$3a(a+3b)(a-3b) = 3a\left(a^2 - 3ab + 3ab - 9b^2\right) = 3a\left(a^2 - 9b^2\right) = 3a^3 - 27ab^2$$

Thus our solution, $3a(a+3b)(a-3b)$ is correct.

b) $100x - x^2 - 2419 =$ $-(x-41)(x-59)$

Solution: We rearrange the polynomial by degree of terms. Since it is quadratic with three terms, it may factor by completing the square. Then we need to factor out -1 to work with a leading coefficient 1 within the parentheses.

$100x - x^2 - 2419 =$

$$\begin{aligned} &= -x^2 + 100x - 2419 \\ &= -\left(x^2 - 100x + 2419\right) \qquad (x-50)^2 = x^2 - 100x + 2500 \\ &= -\left(\underbrace{x^2 - 100x + 2500} - 2500 + 2419\right) \\ &= -\left((x-50)^2 - 81\right) \\ &= -\left((x-50)^2 - 9^2\right) \\ &= -(x-50+9)(x-50-9) \\ &= -(x-41)(x-59) \end{aligned}$$

We check by multiplication:

$$-(x-41)(x-59) = -\left(x^2 - 41x - 59x + 2419\right) = -\left(x^2 - 100x + 2419\right) = -x^2 + 100x - 2419$$

c) $2p^4 - 162 =$ $2\left(p^2+9\right)(p+3)(p-3)$

Solution: We start with the greatest common factor (or GCF).

$$\begin{aligned} 2p^4 - 162 &= \quad \text{factor out GCF} \\ 2\left(p^4 - 81\right) &= \quad \text{re-write both quantities as squares} \\ 2\left(\left(p^2\right)^2 - 9^2\right) &= \quad \text{factor via the difference of squares theorem} \\ 2\left(p^2+9\right)\left(p^2-9\right) &= \quad \text{second factor will factor again} \\ 2\left(p^2+9\right)\left(p^2-3^2\right) &= \quad \text{factor via the difference of squares theorem} \\ &= 2\left(p^2+9\right)(p+3)(p-3) \end{aligned}$$

We check by multiplication:

$$\begin{aligned} 2\left(p^2+9\right)\underbrace{(p+3)(p-3)}_{\text{FOIL}} &= 2\left(p^2+9\right)\left(p^2-3p+3p-9\right) = 2\underbrace{\left(p^2+9\right)\left(p^2-9\right)}_{\text{FOIL}} \\ &= 2\left(p^4-9p^2+9p^2-81\right) = 2\left(p^4-81\right) = 2p^4-162 \end{aligned}$$

Thus our solution, $2\left(p^2+9\right)(p+3)(p-3)$ is correct.

d) $x^2-4x+7=$ does not factor over the real numbers

Solution: We complete the square.

$$\begin{aligned} x^2-4x+7 &= & \qquad (x-2)^2 = x^2-4x+4 \\ \underbrace{x^2-4x+4}-4+7 &= (x-2)^2+3 \end{aligned}$$

This expression can not be factored because the sum of squares can not be factored over the real numbers.

e) $357ab^2-30ab^2x-3ab^2x^2=\ -3ab^2(x+17)(x-7)$

Solution: We start with the GCF (greatest common factor) and rearrange the polynomial by degree of terms.

$$\begin{aligned} 357ab^2-30ab^2x-3ab^2x^2 &= 3ab^2\left(119-x^2-10x\right) \\ &= 3ab^2\left(-x^2-10x+119\right) \end{aligned}$$

The expression is quadratic with three terms, and so it may factor by completing the square. Then we need to factor out -1 to make the leading coefficient 1.

$357ab^2-30ab^2x-3ab^2x^2=$

$$\begin{aligned} &= 3ab^2\left(119-x^2-10x\right) \\ &= -3ab^2\left(x^2+10x-119\right) & \qquad (x+5)^2 = x^2+10x+25 \\ &= -3ab^2\left(\underbrace{x^2+10x+25}-25-119\right) \\ &= -3ab^2\left((x+5)^2-144\right) \\ &= -3ab^2\left((x+5)^2-12^2\right) \\ &= -3ab^2(x+5+12)(x+5-12) \\ &= -3ab^2(x+17)(x-7) \end{aligned}$$

We can check by multiplication.

f) $20x+5x^3=\ 5x\left(x^2+4\right)$

Solution: We rearrange the terms by degree first and then factor out the GCF.

$$20x+5x^3 = 5x^3+20x = 5x\left(x^2+4\right)$$

Since the sum of squares does not factor, the final answer is $5x\left(x^2+4\right)$. We can easily check the result by mulitplication.

2. Solve each of the following equations. Make sure to check your solution.

a) $8x^3 = 50x^2$ 0 and $\frac{25}{4}$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the special zero property.

$$\begin{aligned} 8x^3 &= 50x^2 && \text{subtract } 50x^2 \\ 8x^3 - 50x^2 &= 0 && \text{the GCF is } 2x^2 \\ 2x^2(4x-25) &= 0 \end{aligned}$$

We now apply the special zero property. If this product is zero, then either $2x^2 = 0$ or $4x - 25 = 0$. We solve these equations for x.

$$\begin{aligned} 2x^2 &= 0 && \text{or} && 4x-25=0 \\ 2\cdot x\cdot x &= 0 && \text{or} && 4x = 25 \\ x &= 0 && \text{or} && x = \frac{25}{4} \end{aligned}$$

We check both solutions. If $x = 0$, then

$$\begin{aligned} \text{LHS} &= 8\cdot 0^3 = 8\cdot 0 = 0 \\ \text{RHS} &= 50\cdot 0^2 = 50\cdot 0 = 0 \end{aligned}$$

If $x = \frac{25}{4}$, then

$$\begin{aligned} \text{LHS} &= 8\left(\frac{25}{4}\right)^3 = \frac{8}{1}\cdot\frac{15\,625}{64} = \frac{15\,625}{8} \\ \text{RHS} &= 50\left(\frac{25}{4}\right)^2 = \frac{50}{1}\cdot\frac{625}{16} = \frac{15\,625}{8} \end{aligned}$$

Thus both solutions, 0 and $\frac{25}{4}$ are correct.

b) $8p^3 = 50p$ $-\frac{5}{2}$, 0, and $\frac{5}{2}$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the special zero property.

$$\begin{aligned} 8p^3 &= 50p && \text{subtract } 50p \\ 8p^3 - 50p &= 0 && \text{the GCF is } 2p \\ 2p\left(4p^2-25\right) &= 0 \\ 2p\left((2p)^2 - 5^2\right) &= 0 && \text{factor via difference of squares theorem} \\ 2p(2p+5)(2p-5) &= 0 \end{aligned}$$

We now apply the special zero property. If this product is zero, then either $2p = 0$ or $2p + 5 = 0$ or $2p - 5 = 0$. We solve these equations for p.

$$\begin{aligned} 2p+5 &= 0 && \text{or} && 2p-5=0 && \text{or} && 2p=0 \\ 2p &= -5 && \text{or} && 2p=5 && \text{or} && p=0 \\ p &= -\frac{5}{2} && \text{or} && p=\frac{5}{2} \end{aligned}$$

We check all three solutions. If $p = -\frac{5}{2}$, then

$$\begin{aligned} \text{LHS} &= 8\left(-\frac{5}{2}\right)^3 = \frac{8}{1} \cdot \frac{-125}{8} = -125 \\ \text{RHS} &= 50\left(-\frac{5}{2}\right) = \frac{50}{1} \cdot \frac{-5}{2} = \frac{-250}{2} = -125 \end{aligned}$$

If $p = \frac{5}{2}$, then

$$\begin{aligned} \text{LHS} &= 8\left(\frac{5}{2}\right)^3 = \frac{8}{1} \cdot \frac{125}{8} = 125 \\ \text{RHS} &= 50\left(\frac{5}{2}\right) = \frac{50}{1} \cdot \frac{5}{2} = \frac{250}{2} = 125 \end{aligned}$$

and if $p = 0$, then

$$\begin{aligned} \text{LHS} &= 8 \cdot 0^3 = 8 \cdot 0 = 0 \\ \text{RHS} &= 50 \cdot 0 = 0 \end{aligned}$$

Thus all three solutions, $-\frac{5}{2}$, 0, and $\frac{5}{2}$ are correct.

c) $2x^3 = 20x^2 + 1750x$ $35, 0, -25$

Solution: We reduce one side to zero, then factor, and then apply the zero property.

$$\begin{aligned} 2x^3 &= 20x^2 + 1750x \\ 2x^3 - 20x^2 - 1750x &= 0 && \text{factor out GCF} \\ 2x\left(x^2 - 10x - 875\right) &= 0 && \text{divide both sides by 2} \\ x\left(x^2 - 10x - 875\right) &= 0 \end{aligned}$$

We will factor by completing the square. Half of the linear coefficient is -5, and thus we will work with $(x-5)^2 = x^2 - 10x + 25$. We smuggle in 25.

$$\begin{aligned} x\left(x^2 - 10x - 875\right) &= 0 \\ x\left(\underbrace{x^2 - 10x + 25}-25 - 875\right) &= 0 \\ x\left((x-5)^2 - 900\right) &= 0 \\ x\left((x-5)^2 - 30^2\right) &= 0 \\ x(x-5+30)(x-5-30) &= 0 \\ x(x+25)(x-35) &= 0 \end{aligned}$$

$$x(x+25)(x-35) = 0$$

Applying the zero property we obtain 0, -25, and 35 as the solutions. We check (even if it hurts a little...). If $x = 0$, then

$$\begin{aligned} \text{LHS} &= 2(0)^3 = 0 \\ \text{RHS} &= 20(0)^2 + 1750(0) = 0 \end{aligned}$$

If $x = -25$, then

$$\begin{aligned} \text{LHS} &= 2(-25)^3 = 2(-15\,625) = -31\,250 \\ \text{RHS} &= 20(-25)^2 + 1750(-25) = 20(625) + 1750(-25) = 12\,500 - 43\,750 = -31\,250 \end{aligned}$$

And if $x = 35$, then

$$\begin{aligned} \text{LHS} &= 2(35)^3 = 2(42\,875) = 85\,750 \\ \text{RHS} &= 20(35)^2 + 1750(35) = 20(1225) + 1750(35) = 24\,500 + 61\,250 = 85\,750 \end{aligned}$$

3. Word Problems

a) One number is 18 less than the other. Find these numbers if their product is 1600.
$32, 50$ and $-50, -32$
Solution: Let us denote the smaller number by x. Then the larger number is $x+8$. The equation will express the product of the numbers.

$$\begin{aligned} x(x+18) &= 1600 && \text{multiply} \\ x^2 + 18x &= 1600 && \text{subtract } 1600 \\ x^2 + 18x - 1600 &= 0 && \text{factor by completing the square} \end{aligned}$$

$$\begin{aligned} x^2 + 18x - 1600 &= 0 && (x+9)^2 = x^2 + 18x + 81 \\ \underbrace{x^2 + 18x + 81} - 81 - 1600 &= 0 \\ (x+9)^2 - 1681 &= 0 && \sqrt{1681} = 41 \\ (x+9)^2 - 41^2 &= 0 \\ (x+9+41)(x+9-41) &= 0 \\ (x+50)(x-32) &= 0 \end{aligned}$$

$$x_1 = -50 \qquad x_2 = 32$$

We did not get one pair! We obtained two candidates for the smaller number in two pairs of numbers. If the smaller number is -50, then the larger one is $-50 + 18 = -32$. If the smaller number is 32, then the larger one is $32 + 18 = 50$. It is easy to see that both pairs work:

$$\begin{aligned} -32 - (-50) &= 18 \quad \text{and} \quad -32(-50) = 1600 \\ 50 - 32 &= 18 \quad \text{and} \quad 32(50) = 1600 \end{aligned}$$

Thus there are two solutions, -50 with -32 and 32 with 50.

b) We throw an object upward from the top of a 1200 ft tall building. The height of the object, (measured in feet) t seconds after we threw it is

$$h(t) = -16t^2 + 160t + 1200$$

i) Where is the object 3 seconds after we threw it? 1536 ft
Solution: We need to compute $h(3)$. This means that we substitute 3 into t and evaluate the algebraic expression.

$$\begin{aligned} h(3) &= -16 \cdot 3^2 + 160 \cdot 3 + 1200 = -16 \cdot 9 + 160 \cdot 3 + 1200 \\ &= -144 + 480 + 1200 = 336 + 1200 = 1536 \end{aligned}$$

Thus the object is 1536 ft high after 3 seconds.

ii) How long does it take for the object to hit the ground? 15 seconds.
Solution: we need to solve the equation $t = ?$ so that $h(t) = 0$

$$\begin{aligned} h(t) &= 0 \\ -16t^2 + 160t + 1200 &= 0 && \text{factor out } -16 \\ -16\left(t^2 - 10t - 75\right) &= 0 \end{aligned}$$

We will factor $t^2 - 10t + 75$ by completing the square.

$$\begin{aligned}
-16\left(t^2-10t-75\right) &= 0 && (t-5)^2 = t^2-10t+25 \quad \text{smuggle in } 25\\
-16\left(\underbrace{t^2-10t+25}-25-75\right) &= 0 &&\\
-16\left((t-5)^2-100\right) &= 0 && \text{re-write } 100 \text{ as } 10^2\\
-16\left((t-5)^2-10^2\right) &= 0 && \text{factor via the difference of squares theorem}\\
-16\,(t-5+10)\,(t-5-10) &= 0 && \text{simplify}\\
-16\,(t+5)\,(t-15) &= 0 && \text{apply zero property}
\end{aligned}$$

$$\begin{aligned}
t+5 &= 0 \quad \text{or} \quad t-15=0\\
t &= -5 \quad \text{or} \quad t=15
\end{aligned}$$

Since the negative solution, $t=-5$ does not make sense in the context of the problem, it is ruled out. We check $t=15$:

$$\begin{aligned}
h(3) &= -16\cdot 15^2+160\cdot 15+1200\\
&= -16\cdot 225+160\cdot 15+1200\\
&= -3600+2400+1200\\
&= -1200+1200=0
\end{aligned}$$

Thus the answer is: 15 seconds.

c) Find all numbers that satisfy the following condition: if we square the number, we get back the same number. $0, 1$
Solution: Let us denote the number by x. The equation is

$$\begin{aligned}
x^2 &= x && \text{reduce one side to zero}\\
x^2-x &= 0 && \text{factor}\\
x\,(x-1) &= 0 && \text{apply the zero property}
\end{aligned}$$

$$\begin{aligned}
x &= 0 \quad \text{or} \quad x-1=0\\
x &= 0 \quad \text{or} \quad x=1
\end{aligned}$$

Thus there are two numbers, 0 and 1, satisfying the property. We check: $0^2=0$ and $1^2=1$. Thus our answer is: 0 and 1

d) Find all numbers that satisfy the following condition: if we raise the number to the third power, the result is four times the original number. $0, 2, -2$
Solution: Let us denote the number by x. The equation is

$$\begin{aligned}
x^3 &= 4x && \text{reduce one side to zero}\\
x^3-4x &= 0 && \text{factor out the GCF}\\
x\left(x^2-4\right) &= 0 && \text{factor via the difference of squares theorem}\\
x\,(x+2)\,(x-2) &= 0 && \text{apply the zero property}
\end{aligned}$$

$$\begin{aligned}
x &= 0 \quad \text{or} \quad x+2=0 \quad \text{or} \quad x-2=0\\
x &= 0 \quad \text{or} \quad x=-2 \quad \text{or} \quad x=2
\end{aligned}$$

Thus there are three numbers, 0, 2 and -2, satisfying the property. We check: $0^3 = 4 \cdot 0$, $2^3 = 4 \cdot 2$, and $-2^3 = 4(-2)$. Thus our answer is: 0, 2, and -2.

e) The area of a rectangle is 1260 m^2. Find the dimensions of the rectangle if we know that one side is 48 m longer than three times the other side. 14 m by 90 m

Solution: Let us denote the shorter side by x. Then the longer side is $3x + 48$. We obtain the equation for the area (multiply the two sides):

$$\underbrace{x}_{\text{shorter side}} \underbrace{(3x+48)}_{\text{longer side}} = 1260$$

Since this equation is quadratic, we will reduce one side to zero, and factor the other side to solve the equation.

$$\begin{aligned} x(3x+48) &= 1260 && \text{distribute} \\ 3x^2+48x &= 1260 && \text{subtract } 1260 \\ 3x^2+48x-1260 &= 0 && \text{factor out the GCF, } 3 \\ 3\left(x^2+16x-420\right) &= 0 && \text{divide by } 3 \\ x^2+16x-420 &= 0 && \text{factor} \end{aligned}$$

We will factor by completing the square.

$$\begin{aligned} x^2+16x-420 &= 0 && (x+8)^2 = x^2+16x+64 \\ \underbrace{x^2+16x+64}-64-420 &= 0 \\ (x+8)^2-484 &= 0 \\ (x+8)^2-22^2 &= 0 \\ (x+8+22)(x+8-22) &= 0 \\ (x+30)(x-14) &= 0 \\ x_1 &= -30 \quad \text{and} \quad x_2 = 14 \end{aligned}$$

Since distances can not be negative, $x = -30$ is ruled out. If $x = 14$ m, then the other side is $3(14\,\mathrm{m}) + 48\,\mathrm{m} = 90\,\mathrm{m}$. We check: $90\,\mathrm{m} = 3(14\,\mathrm{m}) + 48\,\mathrm{m}$ and $14\,\mathrm{m}(90\,\mathrm{m}) = 1260\,\mathrm{m}^2$. Thus the rectangle's dimensions are indeed 14 m by 90 m.