Sample Problems

1. Completely factor each of the following.
   a) $4a^2mn - 15abm^2 - 6abmn + 10a^2m^2$
   b) $a^2x^3 - b^2x - a^2x + b^2x^3$
   c) $162a + 162b - 2ax^4 - 2bx^4$
   d) $x^2 - 6x + 8$
   e) $3a^2 - 5a - 2$
   f) $4b^2 - b - 5$
   g) $x^3 - 8y^3$
   h) $125 - 27a^{12}$
   i) $1000 + x^6$
   j) $(x+1)^3 - 27$
   k) $-2a^7 - 2a^4b^9$
   l) $(a+2)^3 + (a-2)^3$
   m) $a^6 - b^6$
2. Solve the equation $8a + 2a^2 = 42$
3. One side of a rectangle is 4 ft shorter than three times the other side. Find the sides if the area is $84\,\text{ft}^2$.

Practice Problems

1. Factor completely.
   a) $30x - 15y + 6ax - 3ay$
   b) $xy - y - x + 1$
   c) $6a^2b^2 - 4a^2bc - 10a^2c^2$
   d) $a^2m + 2a^2n - b^2m - 2b^2n$
   e) $x^2 - 4y^2 + m^2x^2 - 4m^2y^2$
   f) $b^2 - a + ab^2 - 1$
   g) $2m^2 - 18n^4 + 2m^2p^2 - 18n^4p^2$
   h) $a^2x^2 - a^2y^2 + b^2x^2 - b^2y^2$
   i) $3x^2 - 2x - 1$
   j) $6x^2 - 5x + 1$
   k) $14x - 12x^2 + 10$
   l) $5m^2 - 7mn + 2n^2$
   m) $29px - 21p^2 + 10x^2$
   n) $2x^4 - 3y^4 + x^2y^2$
   o) $8x^3 + 1000$
   p) $(q+10)^3 + q^3$
   q) $m^3 - (y+1)^3$
   r) $\left(a^2b\right)\left(a^3 - x^6\right)$
2. Factor completely.
   a) $4a^3m + 2a^3n - 4b^3m - 2b^3n$
   b) $3n^2x^3 - 12m^2y^3 - 12m^2x^3 + 3n^2y^3$
   c) $2a^5 - 2a^2 - 2a^2b^2 + 2a^5b^2$
   d) $(3a-1)^3 + (a-3)^3$
   e) $(3a-1)^3 - (a-3)^3$
   f) $2 - 2x^6$
3. Solve each of the following equations.
   a) $6x^4 - x^3 = 2x^2$
   b) $5a^2 + 5 = 26a$
   c) $11p + 35p^2 = 6$
4. One side of a rectangle is 4 in shorter than 3 times the other side. Find the sides of the rectangle if its area is 319 in^2.

Sample Problems - Answers

1. a) $am(2n+5m)(2a-3b)$ b) $x\left(a^2+b^2\right)(x-1)(x+1)$ c) $2\left(9+x^2\right)(3+x)(3-x)(a+b)$
   d) $(x-2)(x-4)$ e) $(a-2)(3a+1)$ f) $(4b-5)(b+1)$ g) $(x-2y)\left(x^2+2xy+4y^2\right)$
   h) $-\left(3a^4-5\right)\left(9a^8+15a^4+25\right)$ i) $\left(x^2+10\right)\left(x^4-10x^2+100\right)$ j) $(x-2)\left(x^2+5x+13\right)$
   k) $-2a^4\left(a+b^3\right)\left(a^2-ab^3+b^6\right)$ l) $2a\left(a^2+12\right)$ m) $(a+b)(a-b)\left(a^2+ab+b^2\right)\left(a^2-ab+b^2\right)$
2. $-7, 3$
3. 6 ft and 14 ft

Practice Problems - Answers

1. a) $3(a+5)(2x-y)$ b) $(x-1)(y-1)$ c) $2a^2(b+c)(3b-5c)$ d) $(a-b)(a+b)(m+2n)$
   e) $(x-2y)(x+2y)\left(m^2+1\right)$ f) $(b-1)(b+1)(a+1)$ g) $-2\left(3n^2-m\right)\left(3n^2+m\right)\left(p^2+1\right)$
   h) $(x-y)(x+y)\left(a^2+b^2\right)$ i) $(3x+1)(x-1)$ j) $(3x-1)(2x-1)$
   k) $-2(2x+1)(3x-5)$ l) $(5m-2n)(m-n)$ m) $(5x-3p)(2x+7p)$
   n) $(x-y)(x+y)\left(2x^2+3y^2\right)$ o) $8(x+5)\left(x^2-5x+25\right)$ p) $2(q+5)\left(q^2+10q+100\right)$
   q) $(m-y-1)\left(m^2+m+my+y^2+2y+1\right)$ r) $a^2b\left(a-x^2\right)\left(a^2+ax^2+x^4\right)$
2. a) $-2(2m+n)(b-a)\left(a^2+ab+b^2\right)$ b) $3(n-2m)(n+2m)(x+y)\left(x^2-xy+y^2\right)$
   c) $2a^2(a-1)\left(a^2+a+1\right)\left(b^2+1\right)$ d) $4(a-1)\left(7a^2-2a+7\right)$ e) $2(a+1)\left(13a^2-22a+13\right)$
   f) $-2(x-1)(x+1)\left(x+x^2+1\right)\left(x^2-x+1\right)$
3. a) $-\frac{1}{2}, 0, \frac{2}{3}$ b) $\frac{1}{5}, 5$ c) $-\frac{3}{5}, \frac{2}{7}$
4. 11 in by 29 in

Sample Problems - Solutions

1. Completely factor each of the following.

a) $4a^2mn - 15abm^2 - 6abmn + 10a^2m^2$
Solution:

$$\begin{aligned}
4a^2mn - 15abm^2 - 6abmn + 10a^2m^2 &= \quad \text{the GCF is } am \\
am\left(4an - 15bm - 6bn + 10am\right) &= \quad \text{rearrange} \\
am\left(\underbrace{4an - 6bn}\ \underbrace{+10am - 15bm}\right) &= \\
am\left(2n\left(2a-3b\right) + 5m\left(2a-3b\right)\right) &= am\left(2n+5m\right)\left(2a-3b\right)
\end{aligned}$$

The final answer is $am\left(2n+5m\right)\left(2a-3b\right)$. We check by multiplication.

$$\begin{aligned}
am\left(2n+5m\right)\left(2a-3b\right) &= am\left(4an - 6bn + 10am - 15bm\right) \\
&= 4a^2mn - 6abmn + 10a^2m^2 - 15abm^2
\end{aligned}$$

b) $a^2x^3 - b^2x - a^2x + b^2x^3$
Solution:

$$\begin{aligned}
a^2x^3 - b^2x - a^2x + b^2x^3 &= \quad \text{the GCF is } x \\
x\left(a^2x^2 - b^2 - a^2 + b^2x^2\right) &= \quad \text{rearrange} \\
x\left(\underbrace{a^2x^2 - a^2}\ \underbrace{+b^2x^2 - b^2}\right) &= \\
x\left(a^2\left(x^2-1\right) + b^2\left(x^2-1\right)\right) &= x\left(a^2+b^2\right)\left(x^2-1\right)
\end{aligned}$$

We are not done yet since $x^2 - 1 = x^2 - 1^2$ further factors via the difference of squares theorem. The final answer is $x\left(a^2+b^2\right)\left(x+1\right)\left(x-1\right)$. We check by multiplication.

$$\begin{aligned}
x\left(a^2+b^2\right)\left(x^2-1\right) &= x\left(a^2+b^2\right)\left(x^2-1^2\right) \\
&= x\left(a^2+b^2\right)\left(x+1\right)\left(x-1\right)
\end{aligned}$$

c) $162a + 162b - 2ax^4 - 2bx^4$
Solution:

$$\begin{aligned}
162a + 162b - 2ax^4 - 2bx^4 &= \quad \text{the GCF is } 2 \\
2\left(\underbrace{81a + 81b}\ \underbrace{-ax^4 - bx^4}\right) &= \\
2\left(81\left(a+b\right) - x^4\left(a+b\right)\right) &= 2\left(81 - x^4\right)\left(a+b\right)
\end{aligned}$$

We are not done yet, since $81 - x^4 = 9^2 - \left(x^2\right)^2$ further factors via the difference of squares theorem.

$$\begin{aligned}
2\left(81 - x^4\right)\left(a+b\right) &= 2\left(9^2 - \left(x^2\right)^2\right)\left(a+b\right) \\
&= 2\left(9+x^2\right)\left(9-x^2\right)\left(a+b\right)
\end{aligned}$$

One factor still further factors: $9 - x^2 = 3^2 - x^2 = \left(3+x\right)\left(3-x\right)$. Thus the final answer is $2\left(9+x^2\right)\left(3+x\right)\left(3-x\right)(a$
We check by multiplication.

$$\begin{aligned}
&= 2\left(9+x^2\right)\left(9-x^2\right)\left(a+b\right) \\
&= 2\left(9+x^2\right)\left(3^2-x^2\right)\left(a+b\right) \\
&= 2\left(9+x^2\right)\left(3+x\right)\left(3-x\right)\left(a+b\right)
\end{aligned}$$

d) $x^2 - 6x + 8$
Solution: We will factor by grouping. First we conduct the "$pq-$game".
Step 1. Write two equations for pq and $p+q$.

$$\begin{aligned} pq &= 8 & \text{1st coefficient times 3rd coefficient} \\ p+q &= -6 & \text{2nd coefficient} \end{aligned}$$

Step 2. List all possible ways of writing 8 as a product of two natural numbers. There are only two ways,

$$1 \quad 8 \quad \text{and} \quad 2 \quad 4$$

Step 3. Assign negative signs to some numbers in the pairs listed above.
Since the product pq is positive, p and q have to have the same sign.
Since the sum $p+q$ is negative, they both have to be negative.
Thus we only need to consider -1 with -8 and -2 with -4. These pairs all multiply to 8, but their sum will only equal to -6 in one case. Clearly -2 with -4 work as p and q.
Step 4. We use p and q we found to express the second term as the sum of two terms

$$x^2 - 6x + 8 = x^2 - 2x - 4x + 8$$

Step 5. Factor by grouping.

$$\begin{aligned} x^2 - 6x + 8 &= \underbrace{x^2 - 2x}\ \underbrace{-4x + 8} \\ &= x(x-2) - 4(x-2) = (x-4)(x-2) \end{aligned}$$

We check by multiplication:

$$(x-2)(x-4) = x^2 - 4x - 2x + 8 = x^2 - 6x + 8$$

Thus our result is correct.
e) $3a^2 - 5a - 2$
Solution: We will factor by grouping. First we conduct the "$pq-$game".
Step 1. Write two equations for pq and $p+q$.

$$\begin{aligned} pq &= -6 & \text{1st coefficient times 3rd coefficient} \\ p+q &= -5 & \text{2nd coefficient} \end{aligned}$$

Step 2. List all possible ways of writing 6 as a product of two natural numbers. There are only two ways,

$$\begin{matrix} 1 & 6 \\ 2 & 3 \end{matrix}$$

Step 3. Assign negative signs to some numbers in the pairs listed above.
Since the product pq is negative, one of p and q must be negative, and the other must be positive.
Since the sum $p+q$ is negative, we will assign the negative sign to the larger number.
Thus we only need to consider

$$\begin{matrix} 1 & -6 \\ 2 & -3 \end{matrix}$$

These pairs all multiply to -6, but their sum will only equal to -5 in one case. Clearly 1 with -6 work as p and q.
Step 4. We use p and q we found to express the second term as the sum of two terms

$$3a^2 - 5a - 2 = 3a^2 + a - 6a - 2$$

Step 5. Factor by grouping.

$$\begin{aligned} 3a^2 - 5a - 2 &= \underbrace{3a^2 + a}\ \underbrace{-6a - 2} \\ &= a(3a+1) - 2(3a+1) = (a-2)(3a+1) \end{aligned}$$

We check by multiplication:

$$(a-2)(3a+1) = 3a^2 + a - 6a - 2 = 3a^2 - 5a - 2$$

Thus our result is correct.

f) $4b^2 - b - 5$

Solution: we will factor by grouping. First we conduct the "$pq-$game".

$$\begin{aligned} pq &= -20 && \text{1st coefficient times 3rd coefficient} \\ p+q &= -1 && \text{2nd coefficient} \end{aligned}$$

We start by expressing 20 as a product of two numbers. the possible pairs are, 1 with 20, 2 with 10, and 4 with 5. Since the product pq is negative, one number must be positive, the other one must be positive.. Because the sum $p+q$ is negative, the negative sign has to be in front of the larger number. We only need to consider 1 with -20, 2 with -10, and 4 with -5. Clearly 4 with -5 work as p and q. We use these these numbers to express the second term as the sum of two terms, and then factor by grouping.

$$\begin{aligned} 4b^2 - b - 5 &= \underbrace{4b^2 + 4b}\ \underbrace{-5b - 5} \\ &= 4b(b+1) - 5(b+1) = (4b-5)(b+1) \end{aligned}$$

We check by multiplication:

$$(4b-5)(b+1) = 4b^2 + 4b - 5b - 5 = 4b^2 - b - 5$$

Thus our result is correct.

g) $x^3 - 8y^3$

Solution: We will factor via the difference of cubes theorem,
$A^3 - B^3 = (A-B)\left(A^2 + AB + B^2\right)$. In this case, $A = x$ and $B = 2y$.

$$\begin{aligned} x^3 - 8y^3 &= x^3 - (2y)^3 = (x - (2y))\left(x^2 + x(2y) + (2y)^2\right) \\ &= (x-2y)\left(x^2 + 2xy + 4y^2\right) \end{aligned}$$

We check by multiplication:

$$\begin{aligned} (x-2y)\left(x^2 + 2xy + 4y^2\right) &= x\left(x^2 + 2xy + 4y^2\right) - 2y\left(x^2 + 2xy + 4y^2\right) \\ &= x^3 + 2x^2y + 4xy^2 - 2x^2y - 4xy^2 - 8y^3 = x^3 - 8y^3 \end{aligned}$$

h) $125 - 27a^{12}$

Solution: We first factor out -1.

$$125 - 27a^{12} = -\left(27a^{12} - 125\right)$$

We will now factor via the difference of cubes theorem,
$A^3 - B^3 = (A-B)\left(A^2 + AB + B^2\right)$. In this case, $A = 3a^4$ and $B = 5$.

$$\begin{aligned} 125 - 27a^{12} &= -\left(27a^{12} - 125\right) = -\left(\left(3a^4\right)^3 - 5^3\right) \\ &= -\left(\left(3a^4\right) - 5\right)\left(\left(3a^4\right)^2 + \left(3a^4\right)5 + 5^2\right) = -\left(3a^4 - 5\right)\left(9a^8 + 15a^4 + 25\right) \end{aligned}$$

We check by multiplication:

$$\begin{aligned} -\left(3a^4-5\right)\left(9a^8+15a^4+25\right) &= -\left(3a^4\left(9a^8+15a^4+25\right)-5\left(9a^8+15a^4+25\right)\right) \\ &= -\left(27a^{12}+45a^8+75a^4-45a^8-75a^4-125\right) \\ &= -\left(27a^{12}-125\right)=-27a^{12}+125=125-27a^{12} \end{aligned}$$

i) $1000+x^6$
Solution: We will factor via the sum of cubes theorem,
$A^3+B^3=(A+B)\left(A^2-AB+B^2\right)$.

$$\begin{aligned} 1000+x^6 &= x^6+1000=\left(x^2\right)^3+10^3 \\ &= \left(x^2+10\right)\left(\left(x^2\right)^2-10x^2+10^2\right)=\left(x^2+10\right)\left(x^4-10x^2+100\right) \end{aligned}$$

We check by multiplication:

$$\begin{aligned} \left(x^2+10\right)\left(x^4-10x^2+100\right) &= x^2\left(x^4-10x^2+100\right)+10\left(x^4-10x^2+100\right) \\ &= x^6-10x^4+100x^2+10x^4-100x^2+1000 \\ &= x^6+1000=1000+x^6 \end{aligned}$$

j) $(x+1)^3-27$
Solution: We will factor via the difference of cubes theorem,
$A^3-B^3=(A-B)\left(A^2+AB+B^2\right)$. In this case, $A=x+1$ and $B=3$.

$$\begin{aligned} (x+1)^3-27 &= (x+1)^3-3^3=((x+1)-3)\left((x+1)^2+3(x+1)+3^2\right) \\ &= (x-2)\left(x^2+2x+1+3x+3+9\right)=(x-2)\left(x^2+5x+13\right) \end{aligned}$$

k) $-2a^7-2a^4b^9$
Solution:

$$\begin{aligned} -2a^7-2a^4b^9 &= -2a^4\left(a^3+b^9\right)=-2a^4\left(a^3+\left(b^3\right)^3\right) \\ &= -2a^4\left(a+b^3\right)\left(a^2+ab^3+\left(b^3\right)^2\right) \\ &= -2a^4\left(a+b^3\right)\left(a^2-ab^3+b^6\right) \end{aligned}$$

l) $(a+2)^3+(a-2)^3$
Solution: We will factor via the sum of cubes theorem,
$X^3+Y^3=(X+Y)\left(X^2-XY+Y^2\right)$. In this case, $X=a+2$ and $Y=a-2$.

$$\begin{aligned} (a+2)^3+(a-2)^3 &= ((a+2)+(a-2))\left((a+2)^2+(a+2)(a-2)+(a-2)^2\right) \\ &= (a+2+a-2)\left(\left(a^2+4a+4\right)-\left(a^2-4\right)+\left(a^2-4a+4\right)\right) \\ &= (2a)\left(a^2+4a+4-a^2+4+a^2-4a+4\right) \\ &= 2a\left(a^2+12\right)=2a\left(a^2+12\right) \end{aligned}$$

m) a^6-b^6
Solution: We start by the difference of squares theorem.

$$a^6-b^6=\left(a^3+b^3\right)\left(a^3-b^3\right)$$

Now both factors will further factor, via the sum- and difference of cubes theorems. Since

$$a^3+b^3=(a+b)\left(a^2-ab+b^2\right) \quad \text{and} \quad a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

we have that

$$\begin{aligned} a^6 - b^6 &= \left(a^3 + b^3\right)\left(a^3 - b^3\right) \\ &= (a+b)\left(a^2 - ab + b^2\right)(a-b)\left(a^2 + ab + b^2\right) \\ &= (a+b)(a-b)\left(a^2 + ab + b^2\right)\left(a^2 - ab + b^2\right) \end{aligned}$$

2. Solve the equation $8a + 2a^2 = 42$
Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the special zero property.

$$\begin{aligned} 8a + 2a^2 &= 42 && \text{subtract 42, rearrange} \\ 2a^2 + 8a - 42 &= 0 && \text{the GCF is } 2 \\ 2\left(a^2 + 4a - 21\right) &= 0 \end{aligned}$$

We will factor $a^2 + 4a - 21$ by grouping. First we conduct the "$pq-$game".

$$\begin{aligned} pq &= -21 && \text{1st coefficient times 3rd coefficient} \\ p + q &= 4 && \text{2nd coefficient} \end{aligned}$$

We start by expressing 21 as a product of two numbers. the only possible pairs are, 1 with 21 and 3 with 7. Since the product pq is negative, one number must be positive, the other one must be positive. Because the sum $p + q$ is positive, the negative sign has to be in front of the smaller number. We only need to consider -1 with 20, and -3 with 7. Clearly -3 with 7 work as p and q. We use these these numbers to express the second term as the sum of two terms, and then factor by grouping.

$$\begin{aligned} 2\left(a^2 + 4a - 21\right) &= 0 \\ 2\left(\underbrace{a^2 + 7a}\quad\underbrace{-3a - 21}\right) &= 0 \\ 2\left(a(a+7) - 3(a+7)\right) &= 0 \\ 2(a-3)(a+7) &= 0 \end{aligned}$$

Thus our equation is

$$2(a-3)(a+7) = 0$$

We now apply the special zero property. If this product is zero, then either $2 = 0$ or $a - 3 = 0$ or $a + 7 = 0$. We solve these equations for a.

$$\begin{aligned} a - 3 &= 0 && \text{or} && a + 7 = 0 && \text{or} && 2 = 0 \\ a &= 3 && \text{or} && a = -7 && \text{or} && \text{no solution here} \end{aligned}$$

We check both solutions. If $a = 3$, then

$$\text{LHS} = 8(3) + 2(3)^2 = 8 \cdot 3 + 2 \cdot 9 = 24 + 18 = 42 = \text{RHS} \quad \checkmark$$

If $a = -7$, then

$$\text{LHS} = 8(-7) + 2(-7)^2 = 8 \cdot (-7) + 2 \cdot 49 = -56 + 98 = 42 = \text{RHS} \quad \checkmark$$

Thus both solutions, -7 and 3 are correct.

3. One side of a rectangle is 4 ft shorter than three times the other side. Find the sides if the area is $84\,\text{ft}^2$.

Solution: Let us denote the shorter side by x. Then the other side is $3x-4$. The equation expresses the area of the rectangle.

$$\begin{aligned} x(3x-4) &= 84 && \text{multiply out parentheses} \\ 3x^2-4x &= 84 && \text{subtract 84} \\ 3x^2-4x-84 &= 0 \end{aligned}$$

Because the equation is quadratic, we need to factor the left-hand side and then apply the zero property. We will factor by grouping. First we conduct the "pq−game". The sum of p and q has to be the linear coefficient (the number in front of x, with its sign), so it is -4. The product of p and q has to be the product of the other coefficients, $3(-84)=-252$.

$$\begin{aligned} pq &= -252 \\ p+q &= -4 \end{aligned}$$

Now we need to find p and q. Because the product is negative, we re looking for a positive and a negative number. Becuse the sum is negtive, the larger number must carry the negative sign. In summary, we are looking for two numbers that multiply to 252 and differ by 4. (Then we place a negative sign in front of the larger one.) Since these numbers are relatively close to each other, they both must also be close to $\sqrt{252}$. We enter $\sqrt{252}$ into the calculator and obtain decimal:

$$\sqrt{252} \approx 15.874$$

So we start looking for factors of 252, starting at 15, and moving down. We soon find 14 and -18. These are our values for p and q. We use these numbers to express the linear term:

$$-4x = 14x - 18x$$

and factor by grouping.

$$\begin{aligned} 3x^2-4x-84 &= 0 \\ \underbrace{3x^2+14x}\,\underbrace{-18x-84} &= 0 \\ x(3x+14)-6(3x+14) &= 0 \\ (x-6)(3x+14) &= 0 \end{aligned}$$

We now apply the zero property. Either $x-6=0$ or $3x+14=0$. We solve both these equations for x.

$$\begin{aligned} x-6 &= 0 \\ x &= 6 \end{aligned}$$

or

$$\begin{aligned} 3x+14 &= 0 \\ 3x &= -14 \\ x &= -\frac{14}{3} \end{aligned}$$

Since distances can not be negative, the second solution for x, $-\dfrac{14}{3}$ is ruled out. Thus $x=6$. Then the longer side is $3(6\,\text{ft})-4\,\text{ft}=14\,\text{ft}$, and so the rectangle's sides are 6 ft and 14 ft long. We check: $6\,\text{ft}\,(14\,\text{ft}) = 84\,\text{ft}^2$ and $14\,\text{ft} = 3(6\,\text{ft})-4\,\text{ft}$. Thus our solution is correct.

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