When the Unknown Is On Both Sides

We will further study solving linear equations. Let us first recall a few definitions.

Definition: An **equation** is a statement in which two expressions (algebraic or numeric) are connected with an equal sign. A **solution** of an equation is a number (or an ordered set of numbers) that, when substituted into the variable(s) in the equation, makes the statement of equality of the equation true. To **solve an equation** is to find *all* solutions of it. The set of all solutions is also called the solution set.

For example, the equation $-x^2+3=4x-2$ is an equation with two solutions, -5 and 1. We leave it to the reader to verify that these numbers are indeed solution. We will have to deploy systematic methods to find all solutions. The methods we will use usually depends on the type of equation. We will start with the simplest equations, linear equations.

Linear equations are a fundamental concept and tool in mathematics. To solve a linear equation, we isolate the unknown by applying the same operation(s) to both sides. In what follows, we will assume that the reader knows how to solve one-step and two-step linear equations. If you need to review that, please see the handout Linear Equations - Part 1. In what follows, we will solve equationes in which the unknown appears on both sides.

Example 1. Solve the equation $2x-8=5x+10$. Make sure to check your solutions.

Solution: Notice that the unknown appears on both sides. This will be the first thing we will address.

$$\begin{array}{rcll} 2x-8 & = & 5x+10 & \text{subtract } 2x \\ -8 & = & 3x+10 & \text{subtract } 10 \\ -18 & = & 3x & \text{divide by } 3 \\ -6 & = & x & \end{array}$$

The only solution of this equation is -6. We check; if $x=-6$, then the left-hand side (LHS) is

$$\text{LHS} = 2(-6)-8 = -12-8 = -20$$

and the right-hand side (RHS) is

$$\text{RHS} = 5(-6)+10 = -30+10 = -20$$

We find that -6, when substituted into the equation, makes the two sides equal. So our solution, $\boxed{x=-6}$ is correct.

Example 2. Solve the equation $7a-12=-a+20$. Make sure to check your solutions.

Solution:

$$\begin{array}{rcll} 7a-12 & = & -a+20 & \text{add } a \\ 8a-12 & = & 20 & \text{add } 12 \\ 8a & = & 32 & \text{divide by } 8 \\ a & = & 4 & \end{array}$$

So the only solution of this equation is 4. We check; if $a=4$,

$$\text{LHS} = 7\cdot 4-12 = 28-12 = 16 \text{ and } \text{RHS} = -4+20 = 16 \implies \text{LHS} = \text{RHS}$$

So our solution, $\boxed{a=4}$ is correct.

Example 3. Solve the equation $-4x+2=-x+17$. Make sure to check your solutions.

Solution:

$$\begin{aligned} -4x+2 &= -x+17 && \text{add } 4x \\ 2 &= 3x+17 && \text{subtract } 17 \\ -15 &= 3x && \text{divide by } 3 \\ -5 &= x \end{aligned}$$

We check; if $x=-5$, then

$$\text{LHS} = -4(-5)+2 = 20+2 = 22 \text{ and } \text{RHS} = -(-5)+17 = 5+17 = 22 \implies \text{LHS} = \text{RHS}$$

So our solution, $\boxed{x=-5}$ is correct.

Example 4. Solve the equation $\frac{1}{2}m-1=\frac{5}{4}m-\frac{1}{4}$. Make sure to check your solutions.

$$\begin{aligned} \frac{1}{2}m-1 &= \frac{5}{4}m-\frac{1}{4} && \text{subtract } \frac{1}{2}m && \text{margin work: } \frac{5}{4}-\frac{1}{2}=\frac{5}{4}-\frac{2}{4}=\frac{3}{4} \\ -1 &= \frac{3}{4}m-\frac{1}{4} && \text{add } \frac{1}{4} && -1+\frac{1}{4}=\frac{-4}{4}+\frac{1}{4}=-\frac{3}{4} \\ -\frac{3}{4} &= \frac{3}{4}m && \text{divide by } \frac{3}{4} && -\frac{3}{4}\div\frac{3}{4}=-\frac{3}{4}\cdot\frac{4}{3}=-1 \\ -1 &= m \end{aligned}$$

So the only solution of this equation is -1. We check; if $m=-1$,

$$\text{LHS} = \frac{1}{2}(-1)-1 = -\frac{1}{2}-1 = \frac{-1}{2}-\frac{2}{2} = -\frac{3}{2} \text{ and}$$

$$\text{RHS} = \frac{5}{4}(-1)-\frac{1}{4} = -\frac{5}{4}-\frac{1}{4} = -\frac{6}{4} = -\frac{3}{2} \qquad \implies \qquad \text{LHS} = \text{RHS}$$

So our solution, $\boxed{m=-1}$ is correct.

Once the unknown appears on both sides, linear equations might be more complicated. Consider each of the following.

Example 5. Solve each of the following equations. make sure to check your solutions.

a) $5y+16=5y+16$ b) $-6x+5=-6x+9$

Solution:

a) $5y+16=5y+16$

This equation looks different from all the others because the two sides are identical. Logically, the value oftwo sides will be equal no matter what number we substitute into the equation. Computation will confirm this idea.

$$\begin{aligned} 5y+16 &= 5y+16 && \text{subtract } 5y \\ 16 &= 16 \end{aligned}$$

The statement $16=16$ is true, no matter what the value of y is. Such a statement is called an **unconditionally true statement** or **identity**. All numbers are solution of this equation.

b) $-6x+5=-6x+9$

$$\begin{aligned} -6x+5 &= -6x+9 && \text{add } 6x \\ 5 &= 9 \end{aligned}$$

The statement $5=9$ is false, no matter what the value of w is. Such a statement is called an **unconditionally false statement**, or **contradiction**. This equation has no solution.

These situations are new to us. Based on their solution sets, linear equations can be classified as follows.

1. If the last line is of the form $x=5$, the equation is called **conditional**. (This is because the truth value of the statement depends on the value of x. True if x is 5, false otherwise.) A conditional equation has exactly one solution.
2. If the last line is of the form $1=1$, the equation is unconditionally true. Such an equation is called an **identity** and all numbers are solutions of it.
3. If the last line is of the form $3=8$, the equation is unconditionally false. Such an equation is called a **contradiction** and its solution set is the empty set.

Discussion: Classify each of the given equations as a conditional equation, an identity, or a contradiction.

a) $3x+1=3x-1$ b) $2x-4=7x-4$ c) $x-4=4-x$ d) $x-1=-1+x$

Sample Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x+3=4x+9$
2. $3w-5=5w+15$
3. $7x-2=5x-2$
4. $3y-9=-2y+6$
5. $5-x=3x-21$
6. $3a-31=-2a+9$
7. $-2b-6=-9b+8$
8. $3m-1=3m-1$
9. $2w+1=2w-9$
10. $\frac{1}{2}x-1=\frac{2}{3}x+4$
11. $-\frac{2}{5}x+\frac{1}{3}=\frac{4}{15}x$
12. $7x-3=-x+2$

Practice Problems

Solve each of the following equations. Make sure to check your solutions.

1. $5x-8=-4x+1$
2. $-2a+5=7a-13$
3. $-7x=x-24$
4. $-3m-9=2m+16$
5. $1-3x=3x-23$
6. $4p+7=4p-7$
7. $5b-6=-9b-6$
8. $-5y+3=-5y+3$
9. $2x+1=-2x-5$
10. $-\frac{2}{3}x-5=-\frac{1}{2}x+1$
11. $4x-1=x-4$
12. $4x-1=4x+1$
13. $3a+7=3a+7$
14. $\frac{2}{3}x-1=\frac{2}{3}x+1$
15. $\frac{2}{3}x-1=\frac{2}{3}x-1$
16. $\frac{2}{3}x+1=\frac{1}{5}x+1$
17. $\frac{3}{8}x+1\frac{4}{5}=\frac{1}{4}x+1\frac{3}{10}$

Answers

Discussion

a) contradiction b) conditional c) conditional d) identity

Sample Problems

1. -3 2. -10 3. 0 4. 3 5. $\frac{13}{2}$ 6. 8 7. 2 8. identity, all numbers are solution

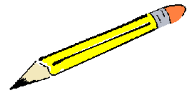
9. there is no solution 10. -30 11. $\frac{1}{2}$ 12. $\frac{5}{8}$

Practice Problems

1. 1 2. 2 3. 3 4. -5 5. 4 6. There is no solution. 7. 0

8. identity, all numers are solution 9. $-\frac{3}{2}$ 10. -36 11. -1 12. There is no solution.

13. identity, all numers are solution 14. There is no solution. 15. identity, all numers are solution 16. 0 17. -4

Sample Problems  Solutions

1. $2x+3=4x+9$

Solution:

$$\begin{aligned} 2x+3 &= 4x+9 && \text{subtract } 2x \text{ from both sides} \\ 3 &= 2x+9 && \text{subtract } 9 \text{ from both sides} \\ -6 &= 2x && \text{divide both sides by 2} \\ -3 &= x \end{aligned}$$

We check: if $x=-3$, then

$$\begin{aligned} \text{LHS} &= 2(-3)+3=-6+3=-3 \\ \text{RHS} &= 4(-3)+9=-12+9=-3 \end{aligned}$$

Thus our solution, $\boxed{x=-3}$ is correct.

2. $3w-5=5w+15$

Solution:

$$\begin{aligned} 3w-5 &= 5w+15 && \text{subtract } 3w \text{ from both sides} \\ -5 &= 2w+15 && \text{subtract } 15 \text{ from both sides} \\ -20 &= 2w && \text{divide both sides by 2} \\ -10 &= w \end{aligned}$$

We check: if $w=-10$, then

$$\begin{aligned} \text{LHS} &= 3(-10)-5=-30-5=-35 \\ \text{RHS} &= 5(-10)+15=-50=15=-35 \end{aligned}$$

Thus our solution, $\boxed{w=-10}$ is correct.

3. $7x-2=5x-2$

Solution:

$$\begin{aligned} 7x-2 &= 5x-2 && \text{subtract } 5x \text{ from both sides} \\ 2x-2 &= -2 && \text{add } 2 \text{ to both sides} \\ 2x &= 0 && \text{divide both sides by 2} \\ x &= 0 \end{aligned}$$

We check: if $x=0$, then

$$\begin{aligned} \text{LHS} &= 7(0)-2=0-2=-2 \\ \text{RHS} &= 5(0)-2=0-2=-2 \end{aligned}$$

Thus our solution, $\boxed{x=0}$ is correct.

4. $3y-9=-2y+6$

   Solution:

$$\begin{aligned} 3y-9 &= -2y+6 && \text{add } 2y \text{ to both sides} \\ 5y-9 &= 6 && \text{add } 9 \text{ to both sides} \\ 5y &= 15 && \text{divide both sides by 5} \\ y &= 3 \end{aligned}$$

   We check: if $y=3$, then

$$\begin{aligned} \text{LHS} &= 3(3)-9=9-9=0 \\ \text{RHS} &= -2(3)+6=-6+6=0 \end{aligned}$$

   Thus our solution, $\boxed{y=3}$ is correct.

5. $5-x=3x-21$

   Solution:

$$\begin{aligned} 5-x &= 3x-21 && \text{add } x \text{ to both sides} \\ 5 &= 4x-21 && \text{add } 21 \text{ to both sides} \\ 26 &= 4x && \text{divide both sides by 5} \\ \frac{26}{4} &= x && \text{reduce result to lowest terms} \\ x &= \frac{13}{2} \end{aligned}$$

   We check: if $x=\frac{13}{2}$, then

$$\begin{aligned} \text{LHS} &= 5-\frac{13}{2}=\frac{10}{2}-\frac{13}{2}=\frac{10-13}{2}=\frac{-3}{2}=-\frac{3}{2} \\ \text{RHS} &= 3\left(\frac{13}{2}\right)-21=\frac{39}{2}-\frac{42}{2}=\frac{-3}{2}=-\frac{3}{2} \end{aligned}$$

   Thus our solution, $\boxed{x=\frac{13}{2}}$ is correct.

6. $3a-31=-2a+9$

   Solution:

$$\begin{aligned} 3a-31 &= -2a+9 && \text{add } 2a \text{ to both sides} \\ 5a-31 &= 9 && \text{add } 31 \text{ to both sides} \\ 5a &= 40 && \text{divide both sides by 5} \\ a &= 8 \end{aligned}$$

   We check: if $a=8$, then

$$\begin{aligned} \text{LHS} &= 3(8)-31=24-31=-7 \\ \text{RHS} &= -2(8)+9=-16+9=-7 \end{aligned}$$

   Thus our solution, $\boxed{a=8}$ is correct.

7. $-2b-6=-9b+8$

Solution:

$$\begin{aligned} -2b-6 &= -9b+8 && \text{add } 9b \text{ to both sides} \\ 7b-6 &= 8 && \text{add } 6 \text{ to both sides} \\ 7b &= 14 && \text{divide both sides by 7} \\ b &= 2 \end{aligned}$$

We check: if $b=2$, then

$$\begin{aligned} \text{LHS} &= -2(2)-6=-4-6=-10 \\ \text{RHS} &= -9(2)+8=-18+8=-10 \end{aligned}$$

Thus our solution, $\boxed{b=2}$ is correct.

8. $3m-1=3m-1$

Solution: This equation looks different from all the others because the two sides are identical. Logically, the two sides will be equal no matter what number we substitute into the equation. Computation will confirm this idea:

$$\begin{aligned} 3m-1 &= 3m-1 && \text{add } 3m \text{ to both sides} \\ -1 &= -1 && \text{add } 1 \text{ to both sides} \\ 0 &= 0 \end{aligned}$$

The statement $0=0$ is true no matter what the value of m is. Such a statement is called an **unconditionally true statement** or **identity**. $\boxed{\textbf{All numbers}}$ **are solutions** of this equation.

9. $2w+1=2w-9$

Solution:

$$\begin{aligned} 2w+1 &= 2w-9 && \text{subtract } 2w \text{ from both sides} \\ 1 &= -9 \end{aligned}$$

The statement $1=-9$ is false no matter what the value of w is. Such a statement is called an **unconditionally false statement**, or **contradiction**. **This equation has** $\boxed{\textbf{no solution}}$**.**

10. $\frac{1}{2}x-1=\frac{2}{3}x+4$

Solution: Structurally, this equation is no different from the previous equations. However, because the coefficients of x are fractions, each step will take a bit more work.

$$\begin{aligned} \frac{1}{2}x-1 &= \frac{2}{3}x+4 && \text{subtract } \frac{1}{2}x \text{ from both sides} \\ -1 &= \frac{1}{6}x+4 && \text{subtract 4 from both sides} \\ -5 &= \frac{1}{6}x && \text{divide both sides by } \frac{1}{6} \\ -30 &= x \end{aligned}$$

Here are the computations for each step. To subtract $\frac{1}{2}x$ from the right-hand side:

$$\frac{2}{3}x-\frac{1}{2}x=\left(\frac{2}{3}-\frac{1}{2}\right)x=\left(\frac{4}{6}-\frac{3}{6}\right)x=\frac{4-3}{6}x=\frac{1}{6}x$$

We divide both sides by $\frac{1}{6}$. To divide is to multiply by the reciprocal:

$$-5 \div \frac{1}{6} = \frac{-5}{1} \div \frac{1}{6} = \frac{-5}{1} \cdot \frac{6}{1} = \frac{-30}{1} = -30$$

We check: if $x = -30$, then

$$\begin{aligned} \text{LHS} &= \frac{1}{2}(-30) - 1 = -15 - 1 = -16 \\ \text{RHS} &= \frac{2}{3}(-30) + 4 = \frac{2}{3} \cdot \frac{-30}{1} + 4 = \frac{-60}{3} + 4 = -20 + 4 = -16 \end{aligned}$$

Thus our solution, $\boxed{x = -30}$ is correct.

11. $-\frac{2}{5}x + \frac{1}{3} = \frac{4}{15}x$

Solution:

$$\begin{aligned} -\frac{2}{5}x + \frac{1}{3} &= \frac{4}{15}x && \text{add } \frac{2}{5}x \text{ to both sides} \\ \frac{1}{3} &= \frac{2}{3}x && \text{divide both sides by } \frac{2}{3} \\ \frac{1}{2} &= x \end{aligned}$$

The computation for each step are as follows. To add $\frac{2}{5}x$ to the right-hand side:

$$\frac{4}{15}x + \frac{2}{5}x = \left(\frac{4}{15} + \frac{2}{5}\right)x = \left(\frac{4}{15} + \frac{6}{15}\right)x = \frac{10}{15}x = \frac{2}{3}x$$

To divide by $\frac{2}{3}$ is to multiply by its reciprocal:

$$\frac{1}{3} \div \frac{2}{3} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

We check: if $x = \frac{1}{2}$, then

$$\begin{aligned} \text{LHS} &= -\frac{2}{5}\left(\frac{1}{2}\right) + \frac{1}{3} = -\frac{1}{5} + \frac{1}{3} = \frac{-3}{15} + \frac{5}{15} = \frac{-3+5}{15} = \frac{2}{15} \\ \text{RHS} &= \frac{4}{15}\left(\frac{1}{2}\right) = \frac{4}{30} = \frac{2}{15} \end{aligned}$$

Thus our solution, $\boxed{x = \frac{1}{2}}$ is correct.

12. $7x - 3 = -x + 2$

Solution: As we will see, this equation is interesting in the sense that solving it will require less work than checking the solution.

$$\begin{aligned} 7x - 3 &= -x + 2 && \text{add } x \text{ to both sides} \\ 8x - 3 &= 2 && \text{add } 3 \text{ to both sides} \\ 8x &= 5 && \text{divide both sides by } 8 \\ x &= \frac{5}{8} && \text{reduce result to lowest terms} \end{aligned}$$

We check: if $x=\frac{5}{8}$, then

$$\begin{aligned} \text{LHS} &= 7\left(\frac{5}{8}\right)-3=\frac{7}{1}\cdot\frac{5}{8}-3=\frac{35}{8}-\frac{24}{8}=\frac{35-24}{8}=\frac{11}{8} \\ \text{RHS} &= -\left(\frac{5}{8}\right)+2=\frac{-5}{8}+\frac{16}{8}=\frac{11}{8} \end{aligned}$$

Thus our solution, $\boxed{x=\frac{5}{8}}$ is correct.

