Part 1 - The Unknown on Both Sides

We will further study solving linear equations. Let us first recall a few definitions.

> **Definition**: An **equation** is a statement in which two expressions (algebraic or numeric) are connected with an equal sign. A **solution** of an equation is a number (or an ordered set of numbers) that, when substituted into the variable(s) in the equation, makes the statement of equality of the equation true. To **solve an equation** is to find *all* solutions of it. The set of all solutions is also called the solution set.

For example, the equation $-x^2+3=4x-2$ is an equation with two solutions, -5 and 1. We leave it to the reader to verify that these numbers are indeed solution. We will have to deploy systematic methods to find all solutions. The methods we will use usually depends on the type of equation. We have been studying the simplest equations, linear equations. We will continue this study with slightly more complex situations in which the unknown appears on both sides of the equation.

Example 1. Solve the equation $2x-8=5x+10$. Make sure to check your solution.

Solution: Recall that in an algebraic expression, the numbers multiplying the unknown are called **coefficients**. In this equation, the coefficients of x are 2 on the left-hand side, and 5 on the rigth-hand side. We can easily reduce this equation to a two-step equation by subtracting $2x$ from both sides.

$$\begin{aligned} 2x-8 &= 5x+10 && \text{subtract } 2x \\ -8 &= 3x+10 && \text{subtract } 10 \\ -18 &= 3x && \text{divide by } 3 \\ -6 &= x \end{aligned}$$

The only solution of this equation is -6. We check; if $x=-6$, then the left-hand side (LHS) is

$$\text{LHS} = 2(-6)-8 = -12-8 = -20$$

and the right-hand side (RHS) is

$$\text{RHS} = 5(-6)+10 = -30+10 = -20$$

So our solution, $\boxed{x=-6}$ is correct.

Note that there is another way to reduce this equation to a two-step equation. Instead of subtracting $2x$, we could also subtract $5x$. Both methods lead to a correct solution. This is a new situation for us. We have a choice between two methods. Although both methods are equally correct, one is better than the other one because it makes the rest of the computation easier. These kind of strategic decisions will become more and more important as we advance in mathematics.

Let us see now, what would happen if we chose to subtract $5x$.

$$\begin{aligned} 2x-8 &= 5x+10 && \text{subtract } 5x \\ -3x-8 &= 10 && \text{add } 8 \\ -3x &= 18 && \text{divide by } -3 \\ x &= -6 \end{aligned}$$

As we can see, now we had to divide by a negative number in the last step. This can be always avoided if we address the side on which the unknown has the smaller coefficient. Keep in mind, coefficients include the sign.

Example 2. Solve the equation $7a-12=-a+20$. Make sure to check your solution.

Solution: Let us first compare the coefficients. On the left-hand side, the coefficient of a is 7. On the right-hand side, the coefficient of a is -1. Since -1 is less than 7, we will eliminate $-a$ from the right-hand side. We reduce this equation to a two-step equation by adding a to both sides.

$$\begin{aligned} 7a-12 &= -a+20 && \text{add } a \\ 8a-12 &= 20 && \text{add } 12 \\ 8a &= 32 && \text{divide by } 8 \\ a &= 4 \end{aligned}$$

So the only solution of this equation is 4. We check; if $a=4$,

$$\text{LHS} = 7\cdot 4 - 12 = 28-12 = 16 \text{ and RHS} = -4+20 = 16 \implies \text{LHS} = \text{RHS}$$

So our solution, $\boxed{a=4}$ is correct.

Example 3. Solve the equation $-4x+2=-x+17$. Make sure to check your solutions.

Solution: -4 is less than -1. Therefore, we will eliminate $-4x$ from the left-hand side.

$$\begin{aligned} -4x+2 &= -x+17 && \text{add } 4x \\ 2 &= 3x+17 && \text{subtract } 17 \\ -15 &= 3x && \text{divide by } 3 \\ -5 &= x \end{aligned}$$

We check; if $x=-5$, then

$$\text{LHS} = -4(-5)+2 = 20+2 = 22 \text{ and RHS} = -(-5)+17 = 5+17 = 22 \implies \text{LHS} = \text{RHS}$$

So our solution, $\boxed{x=-5}$ is correct.

Once the unknown appears on both sides, we might face new situations that were not possible in two-step equations. Consider each of the following.

Example 4. Solve each of the following equations. Make sure to check your solutions.

a) $5y+16=5y+16$ b) $-6x+5=-6x+9$

Solution: a) $5y+16=5y+16$

This equation looks different from all the others because the two sides are identical. Logically, the value of two sides will be equal no matter what number we substitute into the equation. Computation will confirm this idea.

$$\begin{aligned} 5y+16 &= 5y+16 && \text{subtract } 5y \\ 16 &= 16 \end{aligned}$$

The statement $16=16$ is true, no matter what the value of y is. Such a statement is called an **unconditionally true statement** or **identity**. All numbers are solution of this equation.

b) $-6x+5=-6x+9$

$$\begin{aligned} -6x+5 &= -6x+9 \qquad \text{add } 6x \\ 5 &= 9 \end{aligned}$$

When we tried to eliminate the unknown from one side, it disappeared from both sides. We are left with the statement $5=9$. No matter what the value of the unknown is, this statement can not be made true. Indeed, our last line is an **unconditionally false statement**. This means that there is no number that could make this statement true, and so this equation **has no solution**. An equation like this is called a **contradiction**.

These strange situations happen when the unknown has the same coefficient on both sides. Otherwise, there is exactly one solution. Based on their solution sets, linear equations can be classified as follows.

1. If the last line is of the form $x=5$, the equation is called **conditional**. This is because the truth value of the statement depends on the value of x. True if x is 5, false otherwise. A conditional equation has exactly **one solution**.
2. If the last line is of the form $1=1$, the equation is unconditionally true. Such an equation is called an **identity** and **all numbers are solution**s of it.
3. If the last line is of the form $3=8$, the equation is unconditionally false. Such an equation is called a **contradiction** and its solution set is **the empty set**.

Discussion: Classify each of the given equations as a conditional equation, an identity, or a contradiction.
a) $3x+1=3x-1$ b) $2x-4=7x-4$ c) $x-4=4-x$ d) $x-1=-1+x$

Part 2 - Using the Distributive Law

Linear equations might be more complicated. Most often we will be dealing with the distributive law to eliminate parentheses. Then we combine like terms on both sides. After that, these equations will be reduced to the type we just saw.

Example 5. Solve each of the given equations. Make sure to check your solutions.

a) $3x-2(4-x)=3(3x-1)-(x-7)$ b) $4(y-2)-6(3y-5)=5-2(7y+1)$ c) $2(x-2)-(x+6)=x-10$

Solution: a) We first eliminate the parentheses by applying the distributive law.

$$\begin{aligned} 3x-2(4-x) &= 3(3x-1)-(x-7) && \text{eliminate parentheses} && \text{Caution! } -2(-x)=2x \\ 3x-8+2x &= 9x-3-x+7 && \text{combine like terms} && \text{and } -(-7)=7 \\ 5x-8 &= 8x+4 && \text{subtract } 5x \\ -8 &= 3x+4 && \text{subtract } 4 \\ -12 &= 3x && \text{divide by } 3 \\ -4 &= x \end{aligned}$$

We check: if $x=-4$, then

$$\begin{aligned} \text{LHS} &= 3(-4)-2(4-(-4)) = 3(-4)-2\cdot 8 = -12-16 = -28 \quad \text{and} \\ \text{RHS} &= 3(3(-4)-1)-(-4-7) = 3(-12-1)-(-11) = 3(-13)+11 = -39+11 = -28 \quad \Longrightarrow \quad \text{LHS} = \text{RHS} \end{aligned}$$

So our solution, $\boxed{x=-4}$ is correct.

b) We first eliminate the parentheses by applying the distributive law.

$$\begin{aligned} 4(y-2)-6(3y-5) &= 5-2(7y+1) && \text{eliminate parentheses} \\ 4y-8-18y+30 &= 5-14y-2 && \text{combine like terms} \\ -14y+22 &= -14y+3 && \text{add } 14y \\ 22 &= 3 \end{aligned}$$

We are left with the statement $22 = 3$. No matter what the value of the unknown is, this statement can not be made true, this is an **unconditionally false statement**. This equation **has no solution**. An equation like this is called a **contradiction**.

c) We first eliminate the parentheses by applying the distributive law.

$$\begin{aligned} 2(x-2)-(x+6) &= x-10 && \text{eliminate parentheses} \\ 2x-4-x-6 &= x-10 && \text{combine like terms} \\ x-10 &= x-10 && \text{subtract } x \\ -10 &= -10 \end{aligned}$$

When we tried to eliminate the unknown from one side, it disappeared again from both sides. We are left with the statement $-10 = -10$. This statement is true for all values of x. Indeed, our last line is an **unconditionally true statement**. This means that every number makes make this statement true, and so the solution set of this equation is the set of all numbers. An equation like this is called an **identity**.

We often use identities in mathematics, although it seems at first that we would not need equations whose solution set is every number. Consider the following equation: $a+b = b+a$. This equation is an identity, because every pair of numbers is a solution. We use this identity to express a property of *addition*: that the sum of two numbers does not depend on the order of the two numbers.

Part 3 - Applications

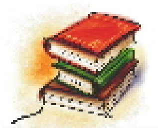
Now that we can solve equations with the unknown on both sides, there are more types of applications that we can solve using linear equations.

Example 6. The sum of three times a number and seven is one greater than twice the sum of eight and the opposite of the number. Find this number.

Solution: We denote the unknown number by x and set up an equation expressing the conditions stated in the equation. The sum of three times a number can be translated as $3x+7$. Twice the sum of eight and the opposite of the number can be translated as $2(8+(-x))$. We can immediately simplify this to $2(8-x)$. (Recall that to subtract is to add the opposite.) The two expressions, $3x+7$ and $2(8-x)$ are being compared: $3x-7$ is one greater than $2(8-x)$. We express this with our equation that we will solve for x.

$$\begin{aligned} 3x+7 &= 2(8-x)+1 && \text{distribute} \\ 3x+7 &= 16-2x+1 && \text{combine like terms} \\ 3x+7 &= -2x+17 && \text{add } 2x \\ 5x+7 &= 17 && \text{subtract } 7 \\ 5x &= 10 && \text{divide by } 5 \\ x &= 2 \end{aligned}$$

We check. Not $x=2$ with the equation, because we might have made a mistake when setting up the equation. Instead, we check if the number 2 works with the conditions stated in the question. The sum of three times two and seven is $3\cdot 2+7$ is 13, and twice the sum of eight and the opposite of two is $2(8+(-2))=2\cdot 6=12$. This works because 13 is indeed one greater than 12. Therefore, our solution is correct.



Sample Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x+3=4x+9$
2. $3w-5=5(w+1)$
3. $7x-2=5x-2$
4. $y-4(y-1)=-y-2(y-2)$
5. $2w+1=2w-9$
6. $4-x=3(x-12)$
7. $7(j-5)+9=2(-2j+5)+5j$
8. $3(x-5)-5(x-1)=-2x+1$

Practice Problems

Solve each of the following equations. Make sure to check your solutions.

9. $5x-3=x+9$
10. $-x+13=2x+1$
11. $-2x+4=5x-10$
12. $5a+1=-2a+1$
13. $5x-7=6x+8$
14. $8x-1=3x+19$
15. $-7x-1=3x-21$
16. $3(x-4)=2(x+5)$
17. $2-3(y-1)=5(y-2)-4(2y-7)$
18. $a-3=5(a-1)-2$
19. $4m-1-3(2m-1)=-4m+8$
20. $3y-2=-2y+18$
21. $8(x-3)-3(5-2x)=x$
22. $5(x-1)-3(x+1)=3x-8$
23. $2(3x-5)-5(2x+1)=-3(x+5)-x$
24. $-2x-(3x-1)=2(5-3x)$
25. $3(x-4)+5(x+8)=2(x-1)$
26. $5(x-1)-3(-x+1)=-3+8x$
27. $2(b+1)-5(b-3)=2(b-7)+1$
28. $3(2x-1)-5(2-x)=4(x-1)+5$
29. $3(2x-7)-2(5x+2)=-5x-30$
30. $3(a-4)-4(a-3)=3(a-2)+2(3-a)$

Answers

Discussion

a) contradiction b) conditional c) conditional d) identity

Sample Problems

1. -3 2. -5 3. 0 4. all real numbers are solution 5. There is no solution. 6. 10 7. 6

8. There is no solution

Practice Problems

1. 3 2. 4 3. 2 4. 0 5. -15 6. 4 7. 2 8. 22 9. No solution 10. 1 11. 3

12. 4 13. 3 14. 0 15. all numbers are soluttion 16. 9 17. -5 18. no solution

19. 6 20. 2 21. -5 22. 0

Sample Problems Solutions

1. $2x+3=4x+9$
Solution:

$$\begin{aligned} 2x+3 &= 4x+9 && \text{subtract } 2x \text{ from both sides} \\ 3 &= 2x+9 && \text{subtract } 9 \text{ from both sides} \\ -6 &= 2x && \text{divide both sides by 2} \\ -3 &= x \end{aligned}$$

We check: if $x=-3$, then

$$\begin{aligned} \text{LHS} &= 2(-3)+3=-6+3=-3 \\ \text{RHS} &= 4(-3)+9=-12+9=-3 \end{aligned}$$

Thus our solution, $\boxed{x=-3}$ is correct. (Note: LHS is short for the left-hand side and RHS is short for the right-hand side.)

2. $3w-5=5(w+1)$
Solution: we first apply the law of distributivity to simplify the right-hand side.

$$\begin{array}{rcll} 3w-5 &=& 5(w+1) & \\ 3w-5 &=& 5w+5 & \text{subtract } 3w \text{ from both sides} \\ -5 &=& 2w+5 & \text{subtract } 5 \text{ from both sides} \\ -10 &=& 2w & \text{divide both sides by 2} \\ -5 &=& w & \end{array}$$

We check. If $w=-5$, then

$$\text{LHS} = 3(-5)-5 = -15-5 = -20 \quad \text{and} \quad \text{RHS} = 5((-5)+1) = 5(-4) = -20$$

Thus our solution, $\boxed{w=-5}$ is correct.

3. $7x-2=5x-2$

Solution:

$$\begin{array}{rcll} 7x-2 &=& 5x-2 & \text{subtract } 5x \text{ from both sides} \\ 2x-2 &=& -2 & \text{add } 2 \text{ to both sides} \\ 2x &=& 0 & \text{divide both sides by 2} \\ x &=& 0 & \end{array}$$

We check: if $x=0$, then

$$\text{LHS} = 7(0)-2 = 0-2 = -2 \quad \text{and} \quad \text{RHS} = 5(0)-2 = 0-2 = -2$$

Thus our solution, $\boxed{x=0}$ is correct.

4. $y-4(y-1)=-y-2(y-2)$

Solution:

$$\begin{array}{rcll} y-4(y-1) &=& -y-2(y-2) & \text{distribute} \\ y-4y+4 &=& -y-2y+4 & \text{combine like terms} \\ -3y+4 &=& -3y+4 & \text{add } 3y \text{ to both sides} \\ 4 &=& 4 & \end{array}$$

This statement is true for all values of y. We also say that the statement is unconditionally true. Such an equation is called an identity, and $\boxed{\text{all real numbers}}$ are solution.

5. $2w+1=2w-9$

Solution:

$$\begin{array}{rcll} 2w+1 &=& 2w-9 & \text{subtract } 2w \text{ from both sides} \\ 1 &=& -9 & \end{array}$$

The statement $1=-9$ is false no matter what the value of w is. Such a statement is called an **unconditionally false statement**, or **contradiction**. **This equation has** $\boxed{\textbf{no solution}}$**.**

6. $4-x=3(x-12)$
Solution: We first apply the law of distributivity to simplify the right-hand side.

$$\begin{aligned} 4-x &= 3(x-12) && \text{distribute } 3 \\ 4-x &= 3x-36 && \text{add } x \text{ to both sides} \\ 4 &= 4x-36 && \text{add } 36 \text{ to both sides} \\ 40 &= 4x && \text{divide both sides by } 4 \\ 10 &= x \end{aligned}$$

We check. If $x=10$, then

$$\text{LHS} = 4-x = 4-10 = -6 \quad \text{and} \quad \text{RHS} = 3(10-12) = 3(-2) = -6$$

Thus our solution, $\boxed{x=10}$ is correct.

7. $7(j-5)+9=2(-2j+5)+5j$
Solution:

$$\begin{aligned} 7(j-5)+9 &= 2(-2j+5)+5j && \text{distribute on both sides} \\ 7j-35+9 &= -4j+10+5j && \text{combine like terms} \\ 7j-26 &= j+10 && \text{subtract } j \\ 6j-26 &= 10 && \text{add } 26 \\ 6j &= 36 && \text{divide by } 6 \\ j &= 6 \end{aligned}$$

We check: if $j=6$, then

$$\begin{aligned} \text{LHS} &= 7(6-5)+9 = 7\cdot 1+9 = 7+9 = 16 \\ \text{RHS} &= 2(-2\cdot 6+5)+5\cdot 6 = 2(-12+5)+30 = 2(-7)+30 = -14+30 = 16 \end{aligned}$$

Thus our solution, $\boxed{j=6}$ is correct.

8. $3(x-5)-5(x-1)=-2x+1$
Solution:

$$\begin{aligned} 3(x-5)-5(x-1) &= -2x+1 && \text{multiply out parentheses} \\ 3x-15-5x+5 &= -2x+1 && \text{combine like terms} \\ -2x-10 &= -2x+1 && \text{add } 2x \\ -10 &= 1 \end{aligned}$$

Since x disappeared from the equation and we are left with an unconditionally false statement, $\boxed{\text{there is no solution}}$ for this equation. This type of an equation is called a **contradiction**.

