Part 1 - One-Step Equations

We will now start solving equations. Recall a few definitons. An **equation** is a statement in which two expressions (algebraic or numeric) are connected with an equal sign. For example, $3x^2 - 5 = x + 28$ is an equation. So is $xy + 5y = -y^2 + x + 2$. A **solution** of an equation is a number (or an ordered set of numbers) that, when substituted into the variable(s) in the equation, makes the statement of equality true. Equations often have more than one solutions. For example, the equation $-3x^2 + x^3 = -x^2 + 5x - 6$ has several solution. We leave to the reader to verify that -2 and 1 are both solutions of this equation. (And, that's not all!)

Definition: To **solve an equation** means to find *all* solutions of it. The set of all solutions of an equation is called the **solution set of the equation**.

If we think about it a little, trial and error is never a legitimate method because there is no way for us to gurantee that there are no other solutions are there. It is impossible for us to try all real numbers because there are infinitely many of them, and we have finite lives.

So we will need to develop systematic methods to solve equations. We will start with the easiest group of equations, linear equations. There are several types of linear equations, and we will start with the easiest type that is called one-step equations.

To solve a linear equation, we isolate the unknown by applying the same operation(s) to both sides. Consider, for example, Ann and Dewitt who has the same monthly salary. This month they both get a 40 dollar raise. Who is making ore money now? It is clear that if we start with two equal quantities and we add the same amount to them, they will still stay equal. This is the underlying principle of solving equations. We always apply the same operations to both sides in an effort to bring the equation in a simple form such as $x = -2$. The following equations are **one-step equations** because there is only one operation that separates us from the desired form.

Example 1. Solve each of the given equations. Make sure to check your solutions.

a) $x - 8 = 10$ b) $3y = -12$ c) $\dfrac{x}{-3} = 8$ d) $m + 10 = -5$

We need to isolate the unknown on one side. In order to do that, we perform the inverse operation. The inverse operation of additon is subtraction and vica versa. The inverse operation of multiplication is division and vica versa.

Solution: a) First we ask: *what happened to the unknown*? On the left-hand side, we see $x - 8$. This means that someone came along and subtracted 8 from the unknown. In order to isolate the unknown x, we need to 'undo' this operation. To do that, we will add 8 to both sides.

$$x - 8 = 10 \qquad \text{add } 8$$
$$x = 18$$

So the only solution of this equation is 18. We can also say that the solution set is $\{18\}$. We should check; if $x = 18$, the left-hand side is

$$\text{LHS} = x - 8 = 18 - 8 = 10 = \text{RHS} \ \checkmark$$

So our solution, $\boxed{x = 18}$ is correct.

b) The unknown was multiplied by 3. In order to isolate the unknown, we will divide both sides by 3.

$$3y = -12 \qquad \text{divide by } 3$$
$$y = -4$$

So the only solution of this equation is -4. We check; if $y = -4$, then

$$\text{LHS} = 3y = 3(-4) = -12 = \text{RHS} \quad \checkmark$$

So our solution, $\boxed{y = -4}$ is correct.

c) The unknown was divided by -3. In order to isolate the unknown, we will reverse division by -3. That is multiplication by -3. As always, we apply the operation to both sides.

$$\frac{x}{-3} = 8 \qquad \text{multiply by } -3$$
$$x = -24$$

So the only solution of this equation is -24. We check; if $x = -24$, then

$$\text{LHS} = \frac{-24}{-3} = 8 = \text{RHS} \quad \checkmark$$

So our solution, $\boxed{x = -24}$ is correct.

d) In order to isolate the unknown, we we subtract 10 from both sides.

$$m + 10 = -5 \qquad \text{subtract } 10$$
$$m = -15$$

So the only solution of this equation is -15. We check; if $m = -15$, then

$$\text{LHS} = m + 10 = -15 + 10 = -5 = \text{RHS} \quad \checkmark$$

So our solution, $\boxed{m = -15}$ is correct.

Why does this work? Consider the equation $x + 3 = 14$. After we subtract three from both sides, we obtain the equation $x = 11$. This statement is true when x is 11 and false otherwise. Because we subtracted three from both sides, that means that the equation $x + 3 = 14$ is also true when x is 11 and false otherwise. In other words, subtracting three from both sides preserved the solution set between the two equations. Such a step is called an equivalent step.

Definition: Performing operations to both sides of an equation in a manner that preserves the solution set is called an **equivalent step.**

Equivalent steps include adding or subtracting the same number from both sides, and any number works. Multiplying or dividing both sides by the same non-zero number are also equivalent steps.

Discussion: Squaring both sides is not an equivalent step. Consider the equation $x = 5$ with solution set $\{5\}$. If we square both sides, we get the equation $x^2 = 25$. What is the solution set of this equation?

Example 2. One side of a rectangle is 12 feet long. Find the length of the other side if the area of the rectangle is 60 square-feet.

Solution: Let us denote the missing side by x. We will write and solve an equation expressing the area of the rectangle.

$$\begin{aligned} 12x &= 60 \qquad \text{divide by 12} \\ x &= 5 \end{aligned}$$

Thus the other side is $\boxed{5\text{ feet}}$ long. Note that if we carry the units in the computation, they will work out perfectly.

$$(12\,\text{ft})\,x = 60\,\text{ft}^2 \qquad \text{divide by } 12\,\text{ft} \qquad \text{margin work:} \quad \frac{60\,\text{ft}^2}{12\,\text{ft}} = 5\frac{\text{ft}\cdot\text{ft}}{\text{ft}} = 5\,\text{ft}$$

$$x = 5\,\text{ft}$$

Sometimes we will solve equations in more abstract forms. The following examples are called **formulas** or **literal equations**. While they might be intimidating for students, the ideas and techniques are the same.

Example 3. Solve each of the given equations for the specified variable.

a) Solve $A+B=C$ for A b) $IR=V$ for I

Solution: a) All of A, B, and C are unknown, but the instructions identify A as the unknown in which we are interested. We preted that we know the values of B and C, we just don't care about them. So, first we ask: *what happened to our unknown*? On the left-hand side, we see $A+B$. This means that someone came along and added B to the unknown A. In order to isolate the unknown A, we need to 'undo' this operation, that is, we will subtract B from both sides. Although we do not know the value of B, we still are subtracting the same amount when subtracting B.

$$\begin{aligned} A+B &= C \qquad \text{subtract } B \\ A &= C-B \end{aligned}$$

What is somewhat unsettling is that the expression $C-B$ does not collapse to a number because their values are not known. Either way, the solution is $\boxed{A=C-B}$. We can still check, if $A=C-B$, then the left-hand side is

$$\text{LHS} = A+B = \underbrace{C-B}_{A}+B = C-B+B = C = \text{RHS} \ \checkmark$$

So our solution, $\boxed{A=C-B}$ is correct.

b) Consider now the equation $IR=V$. This equation is from physics, it is called Ohm's law. If we connect a lightbulb to a battery, the electric current created depends on properties of the light bulb and the battery. Ohm's law expresses the connection between resistance of the lightbulb (denoted by R), the potential or voltage of the battery (denoted by V), and the electric current (denoted by I). Our unknown is I. The unknown was multiplied by R. In order to isolate the unknown, we will divide both sides by R.

$$\begin{aligned} IR &= V \qquad \text{divide by } R \\ I &= \frac{V}{R} \end{aligned}$$

So the only solution of this equation is $\boxed{I=\dfrac{V}{R}}$.

If we can compute with formulas in the abstract, one formula becomes many. We can solve $V=IR$ for I and get $I=\dfrac{V}{R}$ and also, solve $V=IR$ for R and get $R=\dfrac{V}{I}$.

There is an application of one-step equations that helps us with a tricky algebraic expression that comes up often. Suppose we want to express that two numbers add up to 10. If label one number by x, how can we label the other number?

Example 4. Suppose that x represents a number. Let y be another number such that the sum of x and y is 10. Express y in terms of x.

Solution: Our numbers are labeled x and y. We state that their sum is 10 and solve the equation for y in terms of x.

$$x+y=10 \qquad \text{subtract } x$$
$$y=10-x$$

So y can be expressed in terms of x as $\boxed{10-x}$.

Caution! $x-10$ and $10-x$ look similar but they are very different. $x-10$ is a number ten less than x, while $10-x$ is the number, that, when added to x, results in 10. If we confuse the two, we can quickly check which is which by evaluating the expressions using a few numbers for x. We come up with a few values for x, say -10, -5, 1, 6, 10, and 20. Then we evaluate $x-10$ and $10-x$ using these values for x.

x	-10	-5	1	6	10	20
$x-10$	-20	-15	-9	-4	0	10

and

x	-10	-5	1	6	10	20
$10-x$	20	15	9	4	0	-10

We can now easily tell which table's columns add up to 10 and which table has columns in which the second number is ten less than the first one. We needed a few values because sometimes we can get unlucky: notice that if x is 10, then both $10-x$ and $x-10$ give us the same zero.

Part 2 – Two-Step Equations

Suppose we decide to hide a small object, say a coin. We put the coin on the table, then place an envelope over it, and then, just to be sure, we place a hat on top of the envelope. Let us find the coin! To do that, what do we need to remove, and in what order? We would first remove the hat and then the envelope, right?

This is the basis of solving two-step equations. To isolate the unknown, we will perform the inverse operations, in reverse order. What happened last can be undone first.

Example 5. Solve each of the given equations. Make sure to check your solutions.

a) $10=3x-11$ b) $3x+8=-7$ c) $\dfrac{t-7}{2}=-8$ d) $\dfrac{x}{-3}+4=15$

Solution: a) The equation $10=3x-11$ looks unusual in the sense that two-step equations often contain the unknown on the left-hand side. We are always allowed to swap two sides of an equation. If $A=B$, then clearly, also $B=A$. We will do this first. This is an optional step that is always available.

$$10=3x-11 \qquad \text{swap the two sides}$$
$$3x-11=10$$

We now look at the side that contains x and ask: *What happened to the unknown*? The answer is: *Multiplication by* 3 *and then subtraction of* 11. We need to apply the inverse operations, in reverse order. In this case, this means that we will add 11 to both sides and then divide both sides by 3.

$$3x-11=10 \qquad \text{add } 11$$
$$3x=21 \qquad \text{divide by } 3$$
$$x=7$$

So the only solution of this equation is 7. We check: if $x=7$, then

$$\text{LHS}=3x-11=3\cdot 7-11=21-11=10=\text{RHS} \checkmark$$

So our solution, $\boxed{x=7}$ is correct.

b) As we look at the equation $3x+8=-7$ and ask: *What happened to the unknown*? The answer is: *Multiplication by* 3 *and then addition of* 8. We need to apply the inverse operations, in a reverse order. In this case, this means that we will subtract 8 from both sides and then divide both sides by 3.

$$\begin{aligned} 3x+8 &= -7 && \text{subtract } 8 \\ 3x &= -15 && \text{divide by } 3 \\ x &= -5 \end{aligned}$$

So the only solution of this equation is -5. We check: if $x=-5$, then

$$\text{LHS} = 3x+8 = 3(-5)+8 = -15+8 = -7 = \text{RHS} \checkmark$$

So our solution, $\boxed{x=-5}$ is correct.

c) What happened to the unknown? On the left-hand side, there was a subtraction of 7 and then a division by 2. To reverse that, we will multiply both sides by 2 and then add 7 to both sides.

$$\begin{aligned} \frac{t-7}{2} &= -8 && \text{multiply by } 2 \\ t-7 &= -16 && \text{add } 7 \\ t &= -9 \end{aligned}$$

So the only solution of this equation is -9. We check: if $t=-9$, then

$$\text{LHS} = \frac{t-7}{2} = \frac{-9-7}{2} = \frac{-16}{2} = -8 = \text{RHS}\checkmark$$

So our solution, $\boxed{t=-9}$ is correct.

d) What happened to the unknown? On the left-hand side, there was a division by -3 and then an addition of 4. To reverse that, we will subtract 4 from both sides by and then multiply both sides by -3.

$$\begin{aligned} \frac{x}{-3}+4 &= 15 && \text{subtract } 4 \\ \frac{x}{-3} &= 11 && \text{multiply by } -3 \\ x &= -33 \end{aligned}$$

So the only solution of this equation is -33. We check: if $x=-33$, then

$$\text{LHS} = \frac{x}{-3}+4 = \frac{-33}{-3}+4 = 11+4 = 15 = \text{RHS}\checkmark$$

So our solution, $\boxed{x=-33}$ is correct.

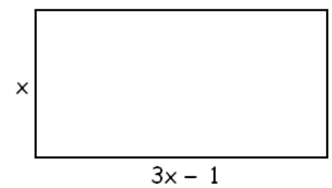
Example 6. The longer side of a rectangle is one feet shorter than three times the shorter side. Find the sides of the rectangle if we also know that its perimeter is 54 feet.

There is no denying that this problem is our first real word problem (also called application problem). The general steps for solving such a problem are as follows.

1. Read the text of the problem. Make a decision about what quantity to label by x. Write this decision down.
2. Read the text of the problem. Label all other quantities in the problem in terms of x. This usually involves translations from English to algebraic expressions using x. Organize all this using a table or a picture.
3. Read the text of the problem. There should be one more piece of information that wasn't used yet. Based on that information, write an equation in x.
4. Solve the equation for x.
5. Now that we know the value of x, all other qualtities in the problem can be found, because they were labeled in terms of x.
6. Read the text of the problem. Formulate your answer.
7. Check whether your solution satisfies all conditions stated in the problem.

At first, do not expect to solve the problem on the first attempt. We often have to try to call several things x before we can successfully solve the problem.

Solution: Let us denote the shorter side by x. Then the longer side can be expressed as $3x-1$. Before we move on, it might be useful to organize our data using a table or a picture.



We will write and solve an equation expressing the perimeter of the rectangle. The perimeter of a rectangle with sides a and b is $P=2a+2b$.

$$2\cdot\underbrace{x}_{\text{shorter side}}+2\cdot\left(\underbrace{3x-1}_{\text{longer side}}\right)=54$$

We will simplify the left-hand side by applying the distributive law and then combining like terms.

$$2x+2(3x-1)=54 \qquad \text{distribute 2}$$
$$2x+6x-2=54 \qquad \text{combine like terms}$$
$$8x-2=54$$

This is now a two-step equation we can easily solve for x.

$$8x-2=54 \qquad \text{add 2}$$
$$8x=56 \qquad \text{divide by 8}$$
$$x=7$$

Now that we found x, we suddenly know everything, because x is 7 everywhere.

shorter side:	x	$=$	7
longer side:	$3x-1$	$=$	$3\cdot 7-1=20$

The shorter side was labeled x. We know now that x is 7. The longer side was labeled $3x-1$. Since x is still 7, that means that the longer side is $3\cdot 7-1=20$. Thus the rectangle has sides 7 feet and 20 feet long.

Good news! We do not need to check if 7 is indeed the solution of the equation. What if we correctly solved the wrong equation? Recall that *we* came up with the equation, it wasn't given. Instead, we should check if our solution satisfies the conditions stated in the problem. Is the longer side indeed one feet shorter than three times the shorter side? Indeed, $3 \cdot 7 - 1 = 20$ ✓ Is the perimeter 54 feet? Indeed,
$2(7\,\text{ft}) + 2(20\,\text{ft}) = 14\,\text{ft} + 40\,\text{ft} = 54\,\text{ft}$. ✓ So the sides are indeed $\boxed{\text{7 feet and 20 feet}}$ long.

Example 7. When Candice took over the register, it contained 30 bills, all five-dollar and ten-dollar bills. How many of each were there if the value of all of these bills together was 210 dollars?

Solution: Suppose that we label the number of ten-dollar bills by x. Then we have $30 - x$ many five-dollar bills. (see Example 4.) We will set up the equation to express the total value of the bills.

	ten-dollar bills	five-dollar bills
number of bills	x	$30 - x$
value in dollars	$10x$	$5(30 - x)$

The equation will express the value of the bills. We solve the equation.

$$\begin{aligned} 10x + 5(30 - x) &= 210 && \text{distribute 5} \\ 10x + 150 - 5x &= 210 && \text{combine like terms} \\ 5x + 150 &= 210 && \text{subtract 150} \\ 5x &= 60 && \text{divide by 5} \\ x &= 12 \end{aligned}$$

Since we labeled the number of ten-dollar bills by x, we have 12 ten-dollar bills. The number of five-dollar bills was labeled $30 - x$, thus there are $30 - 12 = 18$ five-dollar bills. Our answer is:
$\boxed{\text{12 ten-dollar bills and 18 five-dollar bills}}$. We check: $12 + 18 = 30$ so we indeed have 30 bills. The value of all bills is $12 \cdot 10 + 18 \cdot 5 = 120 + 90 = 210$ dollars. Thus our answer is correct.

Example 8. Solve each of the given equations for the unknown indicated.

a) $A = 3B - C$ for B b) $A = 3(B - C)$ for B

Solution: a) We are to solve the equation $A = 3B - C$ for B. Two things happened to the unknown: first a multiplication by 3 and then C was subtracted. To isolate B, we will reverse those operations in the reverse order. This means that we will first add C and then divide by 3.

$$\begin{aligned} A &= 3B - C && \text{add } C \\ A + C &= 3B && \text{divide by 3} \\ \frac{A + C}{3} &= B && \text{and so } \boxed{B = \frac{A + C}{3}}. \end{aligned}$$

b) The equation $A = 3(B - C)$ is very similar to the previous one, because it involves the same two operations; only the order is different. We first subtract C and then multiply by 3. So we will divide by 3 first and then add C.

$$\begin{aligned} A &= 3(B - C) && \text{divide by 3} \\ \frac{A}{3} &= B - C && \text{add } C \\ \frac{A}{3} + C &= B \end{aligned}$$

Thus our solution is $\boxed{B = \frac{A}{3} + C}$.

Most often a presented method is not the only one possible. We can solve this equation differently, by first distributing 3 and then basically solving an equation very similar to the previous example.

$$\begin{aligned} A &= 3(B-C) && \text{distribute 3} \\ A &= 3B-3C && \text{add } 3C \\ A+3C &= 3B && \text{divide by 3} \\ \frac{A+3C}{3} &= B \end{aligned}$$

Both methods are correct, and the two results are the same, although they might appear different at first. Once we have more algebra skills under our belt, we will be able to verify that the two expressions are really the same.

The next example is not a two-step equation but it can be solved using the same method. We will see many steps. We will perform the inverse operations, in the reverse order. It is sort of like an onion we take apart. We can peel of always the outermost layer.

Example 9. Solve the given equation. $\dfrac{\dfrac{\dfrac{3x-1}{5}+2}{-2}-8}{4}=-2$

Solution: The unknown is on the left-hand side, and lots of operations were done to it. In order, there were: multiplication by 3, subtracting 1, division by 5, adding 2, division by -2, subtraction of 8 and division by 4. We will undo them in the reversed order. This means: first multiply by 4, then add 8, then multiply by -2, then subtract 2, then multiply by 5, then add 1, and finally divide by 3. So, that's the plan.

$$\begin{aligned} \frac{\frac{\frac{3x-1}{5}+2}{-2}-8}{4} &= -2 && \text{multiply by 4} \\ \frac{\frac{3x-1}{5}+2}{-2}-8 &= -8 && \text{add 8} \\ \frac{\frac{3x-1}{5}+2}{-2} &= 0 && \text{multiply by } -2 \\ \frac{3x-1}{5}+2 &= 0 && \text{subtract 2} \end{aligned}$$

$$\begin{aligned} \frac{3x-1}{5} &= -2 && \text{multiply by 5} \\ 3x-1 &= -10 && \text{add 1} \\ 3x &= -9 && \text{divide by 3} \\ x &= -3 \end{aligned}$$

We check: if $x=-3$, then the left-hand side is

$$\begin{aligned} \text{LHS} &= \frac{\frac{\frac{3(-3)-1}{5}+2}{-2}-8}{4} = \frac{\frac{\frac{-9-1}{5}+2}{-2}-8}{4} = \frac{\frac{\frac{-10}{5}+2}{-2}-8}{4} = \frac{\frac{-2+2}{-2}-8}{4} = \frac{\frac{0}{-2}-8}{4} \\ &= \frac{0-8}{4} = \frac{-8}{4} = -2 = \text{RHS} \checkmark \end{aligned}$$

Thus our solution, $\boxed{x=-3}$ is correct.

Sample Problems

1. Solve each of the following equations. Make sure to check your solutions.

a) $2x-5=17$ d) $\dfrac{t-5}{12}=4$ g) $\dfrac{x}{3}+8=-2$ j) $3x-10=-10$

b) $\dfrac{a-10}{5}=-3$ e) $2x-7=-3$ h) $-2x+3=3$ k) $-4x+6=-18$

c) $\dfrac{t}{4}-10=-4$ f) $\dfrac{x+8}{3}=-2$ i) $3(x+7)=36$

2. Solve each of the given equations. Make sure to check your solutions.

a) $5(3x-8)-2(8x-1)=-33$

b) $4(5x-3(2x-1)+2x)=52$

c) $2(3(4(5x-1)-17x+2)-3x+1)=14$

d) $\dfrac{\dfrac{\dfrac{5x-1}{7}+3}{5}-10}{3}=-4$

3. Solve each of the given equations for the indicated variables.

a) $2x+y=18$ for y b) $2x+y=18$ for x c) $3y-5x=15$ for y d) $3y-5x=15$ for x

4. Solve each of the following application problems.

a) Ann and Betty dine together. The total bill is \$38. Ann paid \$2 more than Betty. How much did Betty pay?

b) 55 people showed up on the party. There were 3 less women than men. How many men were there?

c) One side of a rectangle is 7 cm shorter than five times the other side. Find the length of the sides if the perimeter of the rectangle is 118 cm.

d) One side of a rectangle is 3 cm longer than four times the other side. Find the sides if the perimeter of the rectangle is 216 cm.

e) The sum of two consecutive even integers is -170. Find these numbers.

f) The sum of three consecutive odd integers is 57. Find these numbers.

g) Small ones weigh 3 lb, big ones weigh 4 lb. The number of small ones is 3 more than twice the number of big ones. All together, they weigh 79 lb. How many small ones are there?

h) We have a jar of coins, all quarters and dimes. All together, they are worth \$17.60 We have 13 more quarters than dimes. How many quarters, how many dimes?

i) Red pens cost \$2 each, blue ones cost \$3 each. We baught some pens. The number of red pens is 7 less than five times the number of blue pens. How many of each did we buy if we paid \$116?

Practice Problems

1. Solve each of the following equations. Make sure to check your solutions.

a) $2x-3=-11$

b) $-2x-3=7$

c) $5x-3=17$

d) $\dfrac{x-3}{7}=-2$

e) $\dfrac{x}{7}-3=-1$

f) $-4x-3=13$

g) $\dfrac{a+1}{4}=-9$

h) $5x-6=-6$

i) $\dfrac{x}{7}-1=-3$

j) $-x+5=-7$

k) $\dfrac{2x-1}{7}=-3$

l) $5(x-2)=-20$

m) $\dfrac{x-8}{7}=-2$

n) $3b+13=-5$

o) $\dfrac{x}{3}-7=7$

2. Solve each of the following equations. Make sure to check your solutions.

a) $3(x+2)-2(5x-1)=-13$

b) $4(2(3(x-5)+7-2x)-x+8)=0$

c) $5(3(3x-1)-2(4x+3))=-50$

d) $-2(5(2x-1)-2(5x+1)-3x)+3=-7$

e) $5x-4(2x-1)-3(x-2)=10$

f) $7x-2((3x+5)-3((4x+1)-5(x-2))-x)=-4$

g) $\dfrac{\dfrac{\dfrac{3x-1}{2}+4}{-5}-6}{2}+7=2$

h) $\dfrac{\dfrac{5\left(\dfrac{5x+2}{-7}-7\right)+11}{2}+12}{5}=2$

3. Solve each of the given equations for the indicated variables. You do not have to check your solutions.

a) $P=2a+2b$ for a

b) $PV=nRT$ for n

c) $2x+3y=-12$ for x

d) $2x+3y=-12$ for y

e) $AB+C=D$ for A

4. Solve each of the following application problems.

a) The sum of two numbers is 27. Their difference is 11. Find these numbers.

b) The sum of two numbers is 11. Their difference is 27. Find these numbers.

c) The sum of two numbers is -11. Their difference is 27. Find these numbers.

d) The sum of two consecutive odd integers is 92. Find these numbers.

e) Julia is 5 years younger than her brother, Tom. How old are they if the sum of their ages is 43?

f) Randy bought four less than three times the number of books that Britney did. Together they bought sixteen books. How many did Britney buy?

g) One side of a rectangle is 6 in shorter than the other side. Find the sides of the rectangle if its perimeter is 120 in.

h) One side of a rectangle is 6 in shorter than twice the other side. Find the sides of the rectangle if its perimeter is 120 in.

i) The largest angle in a triangle is three times as large as the smallest angle. The middle angle is $35°$ larger than the smallest angle. Find the angles in the triangle.

j) A school purchases tickets to a show. A child ticket costs \$8 and an adult ticket costs \$14. The school has paid a total of \$610 for tickets. The number of child tickets was 5 greater than three times the number of adult tickets. How many of the tickets purchased were for adults?

k) The cash register contained 52 bills, all ten-dollar bills and twenty-dollar bills. How many of each was there is the total value of all these bills was 670 dollars?

l) We have a big jar of coins, all quarters and nickels. The number of nickels is three less than twice the number of quarters. How many nickels and quareters are in the jar if the total value of all coins is \$11.75? (Nickels are worth of 5 cents, quarters are worth of 25 cents.)

m) In a hotel, the first night costs 40 dollars, and all additional nights cost 32 dollars. How long did Mr. Williams stay in the hotel if his bill was 520?

Answers

Discussion

1. The solution set of $x^2 = 25$ is $\{5, -5\}$. Both 5 and -5 squares to 25.

Sample Problems

1. a) 11 b) -5 c) 24 d) 53 e) 2 f) -14 g) -30 h) 0 i) 5 j) 0 k) -4

2. a) -5 b) 10 c) 2 d) -18

3. a) $y = 18 - 2x$ b) $x = \dfrac{18 - y}{2}$ c) $y = \dfrac{15 + 5x}{3}$ d) $x = \dfrac{15 - 3y}{-5}$ or $x = \dfrac{3y - 15}{5}$

4. a) Betty paid \$18 and Ann paid \$20. b) There were 26 women and 29 men on the party. c) 11 cm by 48 cm
   d) 21 cm by 87 cm e) -86 and -84 f) 17, 19, and 21 g) 7 big and 17 small
   h) 41 dimes and 54 quarters i) 10 blue and 43 red pens

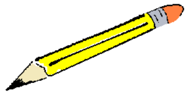
Practice Problems

1. a) -4 b) -5 c) 4 d) -11 e) 14 f) -4 g) -37 h) 0 i) -14 j) 12 k) -10 l) -2
   m) -6 n) -6 o) 42

2. a) 3 b) 8 c) -1 d) -4 e) 0 f) 20 g) 11 h) -6

3. a) $a = \dfrac{P - 2b}{2}$ b) $n = \dfrac{PV}{RT}$ c) $x = \dfrac{-12 - 3y}{2}$ d) $y = \dfrac{-12 - 2x}{3}$ e) $A = \dfrac{D - C}{B}$

4. a) 8 and 19 b) -8 and 19 c) -19 and 8 d) 45 and 47 e) Julia is 19 and Tom is 24
   f) 5 books g) 27 in by 33 in h) 22 in by 38 in i) $29°$, $64°$, and $87°$ j) 15
   k) 37 tens and 15 twenties l) 34 quarters and 65 nickels m) 16 nights



Sample Problems - Solutions

1. Solve each of the following equations. Make sure to check your solutions.

a) $2x - 5 = 17$

Solution: We first ask: *What happened to the unknown?* There was first a multiplication by 2 and then a subtraction of 5. We will undo these operations, in a reverse order. This means first adding 5 and then dividing by 2. As always, of course, we apply all operations to both sides.

$$\begin{aligned} 2x - 5 &= 17 && \text{add 5 to both sides} \\ 2x &= 22 && \text{divide by 2} \\ x &= 11 \end{aligned}$$

We check: if $x = 11$, then

$$\text{RHS} = 2 \cdot 11 - 5 = 22 - 5 = 17 = \text{LHS} \checkmark$$

Thus our solution, $\boxed{x = 11}$ is correct.

b) $\dfrac{a - 10}{5} = -3$

Solution: We first ask: *What happened to the unknown?* There was first a subtraction of 10 and then a division by 5. We will undo these operations in a reverse order. This means first multiplying by 5 and then adding 10. As always, of course, we apply all operations to both sides.

$$\begin{aligned} \frac{a - 10}{5} &= -3 && \text{multiply both sides by 5} \\ a - 10 &= -15 && \text{add } 10 \text{ to both sides} \\ a &= -5 \end{aligned}$$

We check: if $a = -5$, then

$$\text{LHS} = \frac{-5 - 10}{5} = \frac{-15}{5} = -3 = \text{RHS} \checkmark$$

Thus our solution, $\boxed{a = -5}$ is correct.

c) $\dfrac{t}{4} - 10 = -4$

Solution: We first ask: *What happened to the unknown?* There was first a division by 4 and then a subtraction of 10. We will undo these operations in a reverse order. This means first adding 10 and then multiplying by 4. As always, of course, we apply all operations to both sides.

$$\begin{aligned} \frac{t}{4} - 10 &= -4 && \text{add } 10 \text{ to both sides} \\ \frac{t}{4} &= 6 && \text{multiply both sides by 4} \\ t &= 24 \end{aligned}$$

We check: if $t = 24$, then $\text{RHS} = \dfrac{t}{4} - 10 = \dfrac{24}{4} - 10 = 6 - 10 = -4 = \text{LHS} \checkmark$

Thus our solution, $\boxed{t = 24}$ is correct.

d) $\dfrac{t-5}{12}=4$

Solution: We first ask: *What happened to the unknown?* There was first a subtraction of 5 and then a division by 12. We will undo these operations in a reverse order. This means first multiplying by 12 and then adding 5. As always, of course, we apply all operations to both sides.

$$\begin{aligned} \frac{t-5}{12} &= 4 && \text{multiply both sides by 12} \\ t-5 &= 48 && \text{add 5 to both sides} \\ t &= 53 \end{aligned}$$

We check: if $t=53$, then $\text{LHS}=\dfrac{53-5}{12}=\dfrac{48}{12}=4=\text{RHS}$ ✓

Thus our solution, $\boxed{t=53}$ is correct.

e) $2x-7=-3$

Solution: We apply all operations to both sides.

$$\begin{aligned} 2x-7 &= -3 && \text{add } 7 \\ 2x &= 4 && \text{divide by 2} \\ x &= 2 \end{aligned}$$

We check: if $x=2$, then $\text{LHS} = 2\cdot 2-7=4-7=-3=\text{RHS}$ ✓

Thus our solution, $\boxed{x=2}$ is correct.

f) $\dfrac{x+8}{3}=-2$

Solution: We apply all operations to both sides.

$$\begin{aligned} \frac{x+8}{3} &= -2 && \text{multiply by } 3 \\ x+8 &= -6 && \text{subtract } 8 \\ x &= -14 \end{aligned}$$

We check: if $x=-14$, then $\text{LHS} = \dfrac{-14+8}{3}=\dfrac{-6}{3}=-2=\text{ RHS}$ ✓

Thus our solution, $\boxed{x=-14}$ is correct.

g) $\dfrac{x}{3}+8=-2$

Solution: We apply all operations to both sides.

$$\begin{aligned} \frac{x}{3}+8 &= -2 && \text{subtract } 8 \\ \frac{x}{3} &= -10 && \text{multiply by } 3 \\ x &= -30 \end{aligned}$$

We check: if $x=-30$, then

$$\text{LHS}=\frac{-30}{3}+8=-10+8=-2=\text{RHS} ✓$$

Thus our solution, $\boxed{x=-30}$ is correct.

h) $-2x+3=3$

Solution: We apply all operations to both sides.

$$\begin{aligned} -2x+3 &= 3 && \text{subtract } 3 \\ -2x &= 0 && \text{divide by } -2 \\ x &= 0 \end{aligned}$$

We check: if $x=0$, then

$$\text{LHS} = -2 \cdot 0 + 3 = 0 + 3 = 3 = \text{RHS} \checkmark$$

Thus our solution, $\boxed{x=0}$ is correct.

i) $3(x+7)=36$

Solution: We apply all operation to both sides,

$$\begin{aligned} 3(x+7) &= 36 && \text{divide by } 3 \\ x+7 &= 12 && \text{subtract } 7 \\ x &= 5 \end{aligned}$$

We check: if $x=5$, then

$$\text{LHS} = 3(5+7) = 3 \cdot 12 = 36 = \text{RHS} \checkmark$$

Thus our solution, $\boxed{x=5}$ is correct.

j) $3x-10=-10$

Solution:

$$\begin{aligned} 3x-10 &= -10 && \text{add } 10 \text{ to both sides} \\ 3x &= 0 && \text{divide by } 3 \\ x &= 0 \end{aligned}$$

We check: if $x=0$, then

$$\text{LHS} = 3 \cdot 0 - 10 = 0 - 10 = -10 = \text{RHS} \checkmark$$

Thus our solution, $\boxed{x=0}$ is correct.

k) $-4x+6=-18$

Solution:

$$\begin{aligned} -4x+6 &= -18 && \text{subtract } 6 \\ -4x &= -24 && \text{divide by } -4 \\ x &= 6 \end{aligned}$$

We check: if $x=6$, then

$$\text{RHS} = -4x+6 = -4 \cdot 6 + 6 = -24 + 6 = -18 = \text{LHS} \checkmark$$

Thus our solution, $\boxed{x=6}$ is correct.

2. Solve each of the given equations. Make sure to check your solutions.

a) $5(3x-8)-2(8x-1)=-33$

Solution: We need to simplify the left-hand side. After that, the equation will be a simple two-step equation.

$$\begin{aligned} 5(3x-8)-2(8x-1) &= -33 && \text{distribute 5 and } -2\\ 15x-40-16x+2 &= -33 && \text{combine like terms}\\ -x-38 &= -33 && \text{add 38}\\ -x &= 5 && \text{divide (or multiply) by } -1\\ x &= -5 \end{aligned}$$

We check: if $x=-5$, then

$$\begin{aligned} \text{LHS} &= 5[3(-5)-8]-2[8(-5)-1]=5(-15-8)-2(-40-1)=5(-23)-2(-41)\\ &= -115+82=-33=\text{RHS} \checkmark \end{aligned}$$

Thus our solution, $\boxed{x=-5}$, is correct.

b) $4(5x-3(2x-1)+2x)=52$

Solution: We need to simplify the left-hand side. After that, the equation will be a simple two-step equation. We start with the innermost parentheses.

$$\begin{aligned} 4(5x-3(2x-1)+2x) &= 52 && \text{distribute } -3\\ 4(5x-6x+3+2x) &= 52 && \text{combine like terms}\\ 4(x+3) &= 52 && \text{distribute 4}\\ 4x+12 &= 52 && \text{subtract 12}\\ 4x &= 40 && \text{divide by 4}\\ x &= 10 \end{aligned}$$

We check: if $x=10$, then

$$\begin{aligned} \text{LHS} &= 4(5\cdot10-3(2\cdot10-1)+2\cdot10)=4(5\cdot10-3(20-1)+2\cdot10)=4(5\cdot10-3\cdot19+2\cdot10)\\ &= 4(50-57+20)=4(-7+20)=4\cdot13=52=\text{RHS} \checkmark \end{aligned}$$

c) $2(3(4(5x-1)-17x+2)-3x+1)=14$

Solution: After we simplified the expression on the left-had side, we will again have an easy two-step equation. The three paris of parentheses are nested inside each other. We start with the innermost one.

$$\begin{aligned} 2\{3[4(5x-1)-17x+2]-3x+1\} &= 14 && \text{distribute 4}\\ 2\{3(20x-4-17x+2)-3x+1\} &= 14 && \text{combine like terms}\\ 2\{3(3x-2)-3x+1\} &= 14 && \text{distribute 3}\\ 2(9x-6-3x+1) &= 14 && \text{combine like terms} \end{aligned}$$

$$\begin{aligned} 2(6x-5) &= 14 && \text{distribute 2}\\ 12x-10 &= 14 && \text{add 10}\\ 12x &= 24 && \text{divide by 12}\\ x &= 2 \end{aligned}$$

We check: if $x=2$, then

$$\begin{aligned} \text{LHS} &= 2\{3[4(5\cdot2-1)-17\cdot2+2]-3\cdot2+1\}=2(3(4\cdot9-17\cdot2+2)-3\cdot2+1)=2(3(36-34+2)-3\cdot2+1)\\ &= 2(3(2+2)-3\cdot2+1)=2(3\cdot4-3\cdot2+1)=2(12-6+1)=2(6+1)=2\cdot7=14=\text{RHS} \checkmark \end{aligned}$$

So our solution, $\boxed{x=2}$ is correct.

d) $\dfrac{\dfrac{\dfrac{5x-1}{7}+3}{5}-10}{3}=-4$

Soluton: This is one of those many-step equations where we have more than two steps, but we simply perform them in the reverse order like we did in the case of two-step equations.

$$\begin{aligned}
\frac{\frac{\frac{5x-1}{7}+3}{5}-10}{3} &= -4 && \text{multiply by 3}\\
\frac{\frac{5x-1}{7}+3}{5}-10 &= -12 && \text{add 10}\\
\frac{\frac{5x-1}{7}+3}{5} &= -2 && \text{multiply by 5}\\
\frac{5x-1}{7}+3 &= -10 && \text{subtract 3}\\
\frac{5x-1}{7} &= -13 && \text{multiply by 7}\\
5x-1 &= -91 && \text{add 1}\\
5x &= -90 && \text{divide by 5}\\
x &= -18
\end{aligned}$$

We check: if $x=-18$, then

$$\begin{aligned}
\text{LHS} &= \frac{\frac{\frac{5(-18)-1}{7}+3}{5}-10}{3}=\frac{\frac{\frac{-90-1}{7}+3}{5}-10}{3}=\frac{\frac{\frac{-91}{7}+3}{5}-10}{3}=\frac{\frac{-13+3}{5}-10}{3}\\
&= \frac{\frac{-10}{5}-10}{3}=\frac{-2-10}{3}=\frac{-12}{3}=-4=\text{RHS}\checkmark
\end{aligned}$$

Thus our solution, $\boxed{x=-18}$ is correct.

3. Solve each of the given equations for the indicated variables.

a) $2x+y=18$ for y

Solution: We can isolate y by subtracting $2x$ from both sides.

$$2x+y=18 \qquad \text{subtract } 2x$$
$$\boxed{y=18-2x}$$

b) $2x+y=18$ for x

Solution: We can isolate x by subtracting y and then dividing by 2.

$$\begin{aligned}
2x+y &= 18 && \text{subtract } y\\
2x &= 18-y && \text{divide by 2}\\
x &= \frac{18-y}{2}
\end{aligned}$$

So $\boxed{x=\dfrac{18-y}{2}}$.

c) $3y-5x=15$ for y

Solution: We can isolate y by first adding $5x$ to both sides and then by dividing by 3.

$$3y-5x=15 \qquad \text{add } 5x$$

$$3y=15+5x \qquad \text{divide by } 3$$

$$\boxed{y=\frac{15+5x}{3}}$$

d) $3y-5x=15$ for x

We will present two methods to solve this.

Method 1.

$$3y-5x=15 \qquad \text{subtract } 3y$$

$$-5x=15-3y \qquad \text{divide by } -5$$

$$\boxed{x=\frac{15-3y}{-5}}$$

The problem with this method is that in the last step we were forced to divide by a negative number. We will later see that division by a negative number has its dangers. This can be avoided with using the following other method.

Method 2.

$$3y-5x=15 \qquad \text{add } 5x$$

$$3y=15+5x \qquad \text{subtract } 15$$

$$3y-15=5x \qquad \text{divide by } 5$$

$$\boxed{\frac{3y-15}{5}=x}$$

Naturally, the two solutions are the same, but we do not know enough algebra yet to prove that the two expressions are really the same.

4. Solve each of the following application problems.

a) Ann and Betty dine together. The total bill is \$38. Ann paid \$2 more than Betty. How much did Betty pay?

Solution: Let us denote the amount paid by Betty paid by x. Then Ann paid $x+2$. The equation expresses the total amount paid:

$$\begin{aligned} x+x+2 &= 38 && \text{combine like terms} \\ 2x+2 &= 38 && \text{subtract } 2 \\ 2x &= 36 && \text{divide by } 2 \\ x &= 18 \end{aligned}$$

Thus Betty paid \$18 and Ann paid \$20.

b) 55 people showed up on the party. There were 3 less women than men. How many men were there?

Solution: Let us denote the number of women by x. Then $x+3$ men showed up. The equation expresses the total number of people:

$$x+x+3=55 \qquad \text{combine like terms}$$

$$2x+3=55 \qquad \text{subtract } 3$$

$$2x=52 \qquad \text{divide by } 2$$

$$x=26$$

Thus there were 26 women and 29 men on the party.

c) One side of a rectangle is 7 cm shorter than five times the other side. Find the length of the sides if the perimeter of the rectangle is 118 cm.

Solution: Let us denote the shorter side by x. Then the longer side is $5x-7$. We obtain the equation for the perimeter:

$$\begin{aligned} 2x+2(5x-7) &= 118 && \text{distribute} \\ 2x+10x-14 &= 118 && \text{combine like terms} \\ 12x-14 &= 118 && \text{add } 14 \\ 12x &= 132 && \text{divide by } 12 \\ x &= 11 \end{aligned}$$

Thus the shorter side is 11 cm, the longer side is $5(11\,\text{cm})-7\,\text{cm}=48\,\text{cm}$. We check: the perimeter is $2(11\,\text{cm})+2(48\,\text{cm})=118$ cm and 48 is indeed 7 shorter than five times 11. Thus the solution is: $\boxed{11 \text{ cm by } 48 \text{ cm.}}$

d) One side of a rectangle is 3 cm longer than four times the other side. Find the sides if the perimeter of the rectangle is 216 cm.

Solution: Let us denote the shorter side by x. Then the longer side is $4x+3$. We obtain the equation for the perimeter:

$$\begin{aligned} 2x+2(4x+3) &= 216 && \text{distribute} \\ 2x+8x+6 &= 216 && \text{combine like terms} \\ 10x+6 &= 216 && \text{subtract } 6 \\ 10x &= 210 && \text{divide by } 10 \\ x &= 21 \end{aligned}$$

Thus the shorter side is 21 cm, the longer side is $4\cdot 21+3=87$ cm. We check: the perimeter is $2\cdot 21+2\cdot 87=42+174=216\,\text{cm}$ and 87 is indeed 3 longer than four times 21. Thus the solution is: $\boxed{21 \text{ cm by } 87 \text{ cm}}$.

e) The sum of two consecutive even integers is -170. Find these numbers.

Solution: Let us denote the smaller number by x. Then the larger number is $x+2$. The equation expresses the sum of the numbers.

$$\begin{aligned} x+x+2 &= -170 && \text{combine like terms} \\ 2x+2 &= -170 && \text{subtract } 2 \\ 2x &= -172 && \text{divide by } 2 \\ x &= -86 \end{aligned}$$

Then the larger number must be $-86+2=-84$. Thus the numbers are are $\boxed{-86 \text{ and } -84}$.

f) The sum of three consecutive odd integers is 57. Find these numbers.

Solution: Let us denote the smallest number by x. Then the other two numbers must be $x+2$ and $x+4$. The equation expresses the sum of the three numbers.

$$\begin{aligned} x+x+2+x+4 &= 57 && \text{combine like terms} \\ 3x+6 &= 57 && \text{subtract } 6 \\ x &= 51 && \text{divide by } 3 \\ x &= 17 \end{aligned}$$

Thus the three numbers are 17, and $17+2=19$, and $17+4=21$. We check: indeed, $17+19+21=57$. Thus the solution is $\boxed{17,\ 19, \text{ and } 21}$.

g) Small ones weigh 3 lb, big ones weigh 4 lb. The number of small ones is 3 more than twice the number of big ones. All together, they weigh 79 lb. How many small ones are there?

Solution: Let us denote the number of big ones by x. Then the number of small ones is $2x+3$. We obtain the equation expressing the total weight:

$$\begin{aligned} 3(2x+3)+4x &= 79 && \text{distribute} \\ 6x+9+4x &= 79 && \text{combine like terms} \\ 10x+9 &= 79 && \text{subtract 9} \\ 10x &= 70 && \text{divide by 10} \\ x &= 7 \end{aligned}$$

The number of big ones is then 7, and so the number of small ones is $2(7)+3=17$. We check: the number of small ones, 17 is indeed 3 more than twice the number of big ones, 7. The total weight is $7(4)+17(3)=28+51=79$. Thus the solution is $\boxed{\text{7 big, 17 small}}$.

h) We have a jar of coins, all quarters and dimes. All together, they are worth \$17.60 We have 13 more quarters than dimes. How many quarters, how many dimes?

Solution: Let us denote the number of dimes by x. Then the number of quarters must be $x+13$. We obtain the equation by expressing the total value, **in pennies**.

$$\begin{aligned} 10x+25(x+13) &= 1760 && \text{distribute} \\ 10x+25x+325 &= 1760 && \text{combine like terms} \\ 35x+325 &= 1760 && \text{subtract 325} \\ 35x &= 1435 && \text{divide by 35} \\ x &= 41 \end{aligned}$$

Thus we have 41 dimes and $41+13=54$ quarters. We check: $41(0.10)+54(0.25)=4.10+13.50=17.6$. Thus the solution is $\boxed{\text{41 dimes and 54 quarters}}$.

i) Red pens cost \$2 each, blue ones cost \$3 each. We baught some pens. The number of red pens is 7 less than five times the number of blue pens. How many of each did we buy if we paid \$116?

Solution: Let us denote the number of blue pens by x. Then the number of red pens is $5x-7$. The equation will express the total cost of the pens:

$$\begin{aligned} 2(5x-7)+3(x) &= 116 && \text{distribute} \\ 10x-14+3x &= 116 && \text{combine like terms} \\ 13x-14 &= 116 && \text{add 14} \\ 13x &= 130 && \text{divide by 13} \\ x &= 10 \end{aligned}$$

Thus we baught 10 blue and $5(10)-7=43$ red pens. We check:

$$\begin{aligned} 43 &= 5(10)-7 \checkmark \\ 2(43)+3(10) &= 86+30=116 \checkmark \end{aligned}$$

Thus our solution is correct; we baught $\boxed{\text{10 blue and 43 red pens}}$.

