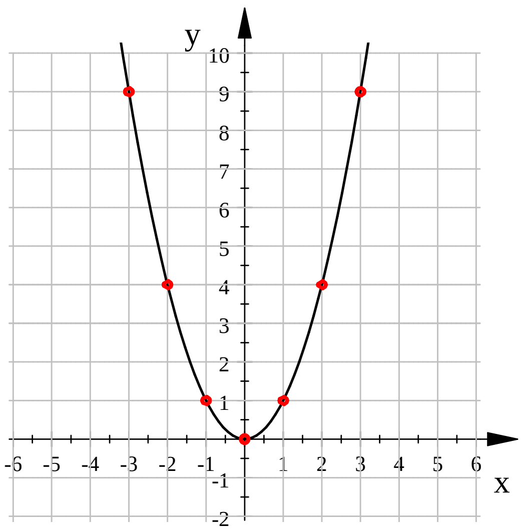
We have been graphing equations in x, y, or x and y. Until now, we have only graphed linear equations - and the graph was a line. We will see that the graphs of quadratic equations are very different.

Example 1. Graph the equation $y = x^2$.

Solution: Just like in case of quadratic equations, we can find points on the graph by selecting a value for x and comuting the y belonging to it using the equation $y = x^2$. We collect the points we found in a table and connect the points.

x	y
0	0
1	1
2	4
3	9
-1	1
-2	4
-3	9



We can always obtain more points. For example, if $x = \frac{1}{2}$, then $y = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Therefore, the point $\left(\frac{1}{2}, \frac{1}{4}\right)$ is also on the graph.

The shape we obtain is called a **parabola**. If the equation islinear in y and quadratic in x, the shape of its graph will be a parabola. Notice that there are significant differences from lines.

There is no point below the $x-$axis. This is because there is no point on the parabole with a negative $y-$value. This is because $y = x^2$ and squares can never be negative.

There is a lowest point. As we have discussed before, quadratic expressions have a smallest possible value. This is because zero can be obtained as the square of zero, but no square is a negative number. The lowest point on the parabola is called the **vertex**.

There is a new symmetry we observe that was not present in lines. The parabola $y = x^2$ is symmetrical to the $y-$axis. This is because a number and its opposite have the same square. Therefore, the $y-$value assigned to 2 is the same as the $y-$value assigned to -2.

The parabola is not straight, it is curved. As the value of x is increased by 1, the $y-$values no longer increase by the same amount, causing the graph to be curved. Notice that as x increases by 1 from 0 to 1 to 2 to 3, the corresponding $y-$values increase by more and more. The differences are 0, 1, 3, 5, and so on. Most quadratic equations will have slightly more complicated graphs than that of $y = x^2$.

Example 2. Graph the parabola $y = x^2 - 8x + 7$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution: We start with algebra. We obtain all three form of the equation first. These three forms are the polynomial form, the standard form, and the factored form. The polynomial form was given.

$$y = x^2 - 8x + 7 \implies \textbf{polynomial form}$$

We then complete the square. Half of the linear coefficient is -4, thus we work out $(x-4)^2$.

$$(x-4)^2 = (x-4)(x-4) = x^2 - 4x - 4x + 16 = x^2 - 8x + 16$$

We are now ready to complete the square:

$$\begin{aligned} y &= x^2 - 8x + 7 && (x-4)^2 = x^2 - 8x + 16 \\ y &= \underbrace{x^2 - 8x + 16} - 16 + 7 \\ y &= (x-4)^2 - 9 \implies \textbf{standard form} \end{aligned}$$

We factor via the difference of squares theorem

$$\begin{aligned} y &= (x-4)^2 - 3^2 \quad \text{since } 3^2 = 9 \\ y &= (x-4+3)(x-4-3) \\ y &= (x-1)(x-7) \implies \textbf{factored form} \end{aligned}$$

From the polynomial form, we easily obtain the $y-$intercept, $(0,7)$, since

$$\text{If } x = 0, \text{ then } y = 0^2 - 8(0) + 7 = 7 \implies \text{found } (0,7)$$

From the standard form, we obtain the coordinates of the vertex.

$$y = (x-4)^2 - 9$$

The trick is to think of the vertex as the point on the graph where the quadratic expression achieves its lowest possible value. Let us start with $(x-4)^2$. Because it is a square, $(x-4)^2$ will be non-negative for every value of x. The lowest possible value of $(x-4)^2$ is zero, when we square zero. This means that $x - 4 = 0$ must hold, thus $x = 4$.

In short, the lowest value of $(x-4)^2$ is zero, when $x = 4$.

Consequently, the lowest possible value of $(x-4)^2 - 9$ is -9, when $x = 4$. Thus the coordinates of the vertex are $(4,-9)$.

The factored form tells us the coordinates of the $x-$intercepts. For the $x-$intercepts, we have to solve the equation

$$\begin{aligned} x &= ? \text{ so that } y = 0 \\ x &= ? \text{ so that } x^2 - 8x + 7 = 0 \end{aligned}$$

$$\begin{aligned} 0 &= x^2 - 8x + 7 \\ 0 &= (x-1)(x-7) \implies x_1 = 1 \quad x_2 = 7 \end{aligned}$$

Thus $y = 0$ when $x = 1$ and $x = 7$. Thus, there are two $x-$intercepts, $(1,0)$ and $(7,0)$.

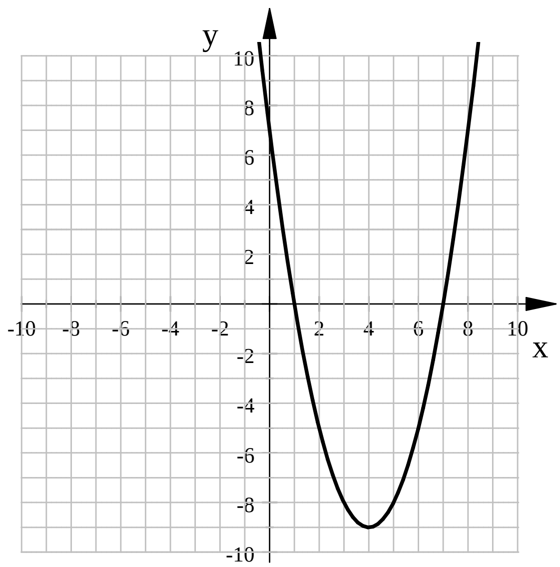
We will compute a few more points before graphing the parabola. To avoid finding points with too large coordinates, we will work with $x-$coordinates, close to the vertex. The $y-$coordinate can be found by substituting values for x into any of the three forms of the equations to find y. This time we will work with the polynomial form.

$$\text{if } x = 2, \text{ then } y = (2)^2 - 8(2) + 7 = 4 - 16 + 7 = -5 \implies \text{found } (2,-5)$$

$$\text{if } x = 3, \text{ then } y = (3)^2 - 8(3) + 7 = 9 - 24 + 7 = -8 \implies \text{found } (3, -8)$$
$$\text{if } x = 4, \text{ then } y = -9 \text{ already found } (4, -9)$$
$$\text{if } x = 5, \text{ then } y = (5)^2 - 8(5) + 7 = 25 - 40 + 7 = -8 \implies \text{found } (5, -8)$$
$$\text{if } x = 6, \text{ then } y = (6)^2 - 8(6) + 7 = 36 - 48 + 7 = -5 \implies \text{found } (6, -5)$$

We are ready to graph: we have the following points, listed left to right.

$y-$intercept	$(0,7)$
$x-$intercept	$(1,0)$
	$(2,-5)$
	$(3,-8)$
vertex	$(4,-9)$
	$(5,-8)$
	$(6,-5)$
$x-$intercept	$(7,0)$



Example 3. Graph the parabola $y = x^2 + 6x + 5$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution: We first obtain all three forms of the equation.

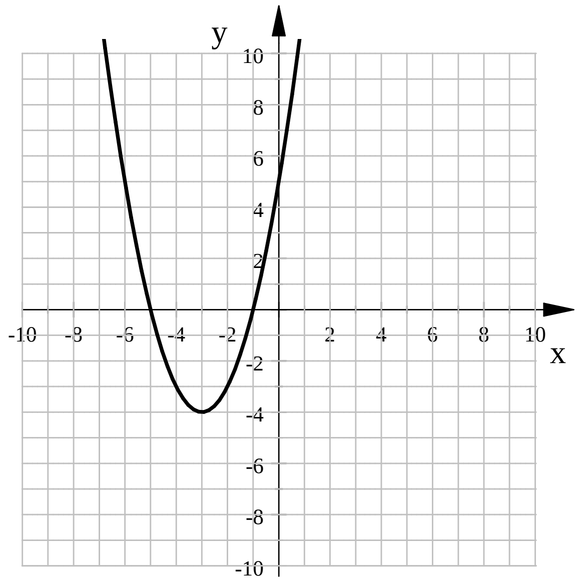
$$\begin{aligned}
y &= x^2 + 6x + 5 \implies \textbf{polynomial form} \implies y - \text{intercept: } (0,5) \\
y &= x^2 + 6x + 5 \qquad\qquad (x+3)^2 = x^2 + 6x + 9 \\
y &= \underbrace{x^2 + 6x + 9} - 9 + 5 \\
y &= (x+3)^2 - 4 \implies \textbf{standard form} \implies \text{vertex: } (-3,-4) \\
y &= (x+3)^2 - 2^2 \\
y &= (x+3+2)(x+3-2) \\
y &= (x+5)(x+1) \implies \textbf{factored form} \implies x - \text{intercepts } (-5,0), \ (-1,0)
\end{aligned}$$

We find a few points close to the vertex, (i.e. x is close to -3). This time we will use the standard square form of the equation, $y = (x+3)^2 - 4$.

$$\begin{aligned}
\text{if } x &= -5, \text{ then } y = 0 \text{ already found } (-5,0) \\
\text{if } x &= -4, \text{ then } y = (-4+3)^2 - 4 = (-1)^2 - 4 = 1 - 4 = -3 \implies \text{found } (-4,-3) \\
\text{if } x &= -3, \text{ then } y = -4 \text{ already found } (-3,-4) \\
\text{if } x &= -2, \text{ then } y = (-2+3)^2 - 4 = 1^2 - 4 = 1 - 4 = -3 \implies \text{found } (-2,-3) \\
\text{if } x &= -1, \text{ then } y = 0 \text{ already found } (-1,0)
\end{aligned}$$

We are ready to graph: we have the following points, listed left to right:

$x-$intercept	$(-5,0)$
	$(-4,-3)$
vertex	$(-3,-4)$
	$(-2,-3)$
$x-$intercept	$(-1,0)$
$y-$intercept	$(0,5)$



Example 4. Graph the parabola $y = x^2 - 2x - 15$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

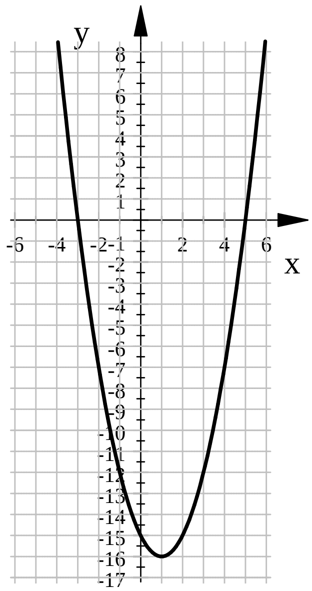
Solution: We first obtain all three forms of the equation.

$$
\begin{aligned}
y &= x^2 - 2x - 15 && \Longrightarrow \ y - \text{intercept: } (0,-15) \\
y &= x^2 - 2x - 15 && \qquad (x-1)^2 = x^2 - 2x + 1 \\
y &= \underbrace{x^2 - 2x + 1} - 1 - 15 \\
y &= (x-1)^2 - 16 && \Longrightarrow \ \text{vertex: } (1,-16) \\
y &= (x-1)^2 - 4^2 \\
y &= (x-1+4)(x-1-4) \\
y &= (x+3)(x-5) && \Longrightarrow \ x - \text{intercepts } (-3,0), \ (5,0)
\end{aligned}
$$

We find a few points close to the vertex, (i.e. x is close to -3). This time we will use the factored form of the equation, $y = (x+3)(x-5)$.

$$
\begin{aligned}
& && (-3,0) \quad \text{already found} \\
\text{if } x &= && -1, \text{ then } y = (-1+3)(-1-5) = 2(-6) = -12 \quad \Longrightarrow \quad (-1,-12) \\
& && (0,-15) \quad \text{already found} \\
& && (1,-16) \quad \text{already found} \\
\text{if } x &= && 2, \text{ then } y = (2+3)(2-5) = 5(-3) = -15 \quad \Longrightarrow \quad (2,-15) \\
\text{if } x &= && 3, \text{ then } y = (3+3)(3-5) = 6(-2) = -12 \quad \Longrightarrow \quad (3,-12) \\
& && (5,0) \quad \text{already found}
\end{aligned}
$$

additional points can be similary found: $(-4,9)$ and $(6,9)$.



Practice Problems

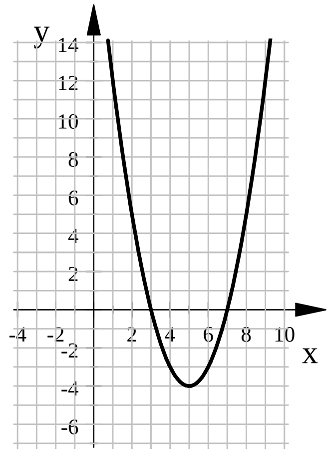
Graph each of the parabolas given below. In each case, state the coordinates of five points of the parabola, including vertex and intercepts.

1. $y = x^2 - 10x + 21$
2. $y = x^2 + 4x - 5$
3. $y = x^2 - 4x + 3$

Answers

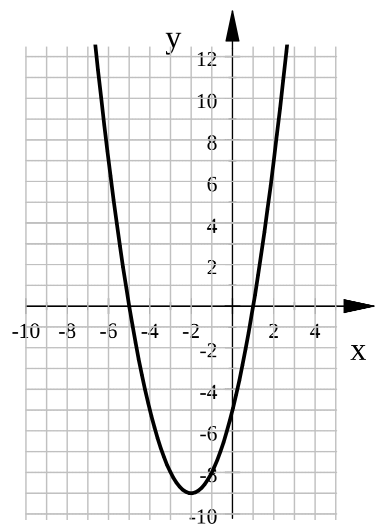
1. $y = x^2 - 10x + 21$

$$\begin{aligned} y &= x^2 - 10x + 21 \implies y - \text{intercept: } (0, 21) \\ y &= (x-5)^2 - 4 \implies \text{vertex: } (5, -4) \\ y &= (x-3)(x-7) \implies x - \text{intercepts: } (3,0) \text{ and } (7,0) \end{aligned}$$



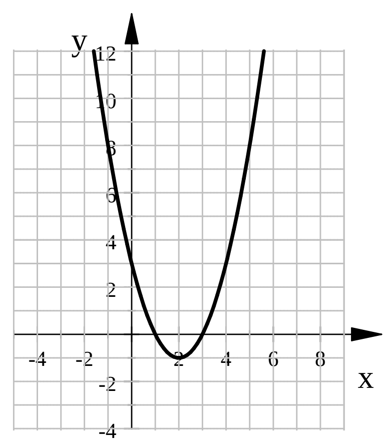
2. $y = x^2 + 4x - 5$

$$\begin{aligned} y &= x^2 + 4x - 5 \implies y - \text{intercept: } (0, -5) \\ y &= (x+2)^2 - 9 \implies \text{vertex: } (-2, -9) \\ y &= (x+5)(x-1) \implies x - \text{intercepts: } (-5,0) \text{ and } (1,0) \end{aligned}$$



3. $y = x^2 - 4x + 3$

$$\begin{aligned} y &= x^2 - 4x + 3 \quad y - \text{intercept: } (0, 3) \\ y &= (x-2)^2 - 1 \quad \text{vertex: } (2, -1) \\ y &= (x-1)(x-3) \quad x - \text{intercepts: } (1,0) \text{ and } (3,0) \end{aligned}$$



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