Example 1. Graph the parabola $y = 20x - 5x^2 + 60$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution: We arrange the terms by degree to obtain the **polynomial form,** $y = -5x^2 + 20x + 60$. From the polynomial form, we obtain the $y-$intercept by substituting $x = 0$. If $x = 0$, then $y = -5 \cdot 0^2 + 20 \cdot 0 + 60 = 60$. And so the $y-$intercept is $(0, 60)$. We factor out the leading coefficient and complete the square to obtain the standard form of the equation.

$$\begin{aligned}
y &= -5x^2 + 20x + 60 \\
y &= -5\left(x^2 - 4x - 12\right) && (x-2)^2 = x^2 - 4x + 4 \\
y &= -5\left(\underbrace{x^2 - 4x + 4} - 4 - 12\right) \\
y &= -5\left((x-2)^2 - 16\right) && \text{distribute } -5 \\
y &= -5(x-2)^2 + 80
\end{aligned}$$

The equation $y = -5(x-2)^2 + 80$ is the **standard form** of the equation of the parabola. It is the form we use to determine the vertex of the parabola. The vertex has to do with the complete square being zero. For the complete square to be zero, we solve

$$\begin{aligned}
(x-2)^2 &= 0 \\
x - 2 &= 0 \\
x &= 2
\end{aligned}$$

And so the $x-$coordinate of the vertex is 2. And if the complete square is zero, then the $y-$coordinate of the vertex can be easily found:

$$y = -5(x-2)^2 + 80 = -5 \cdot 0 + 80 = 80$$

so the vertex is $(2, 80)$.

Notice that the expression $-5(x-2)^2$ is always negative or zero, and so now 80 is the greatest value that the expression $-5(x-2)^2 + 80$ achieves. Indeed, **if the leading coefficient is negative, the parabola opens downward**. We now factor the expression to find the coordinates of the $x-$intercepts. For the $x-$intercepts, we solve the equation

$$\begin{aligned}
x &= ? \quad \text{so that} \quad y = 0 \\
x &= ? \quad \text{so that} \quad -5x^2 + 20x + 60 = 0
\end{aligned}$$

We continue the computation from the second last line and factor via the difference of squares theorem.

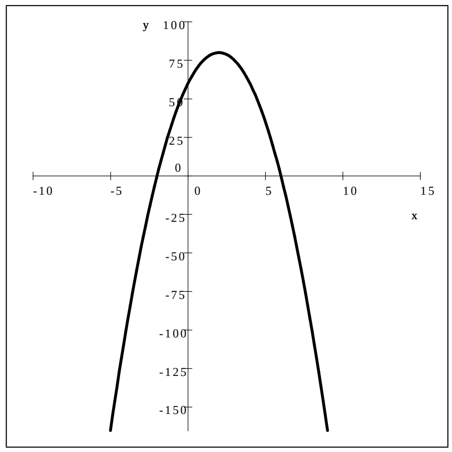
$$\begin{aligned}
-5\left((x-2)^2 - 16\right) &= 0 \\
-5\left((x-2)^2 - 4^2\right) &= 0 \\
-5(x-2+4)(x-2-4) &= 0 \\
-5(x+2)(x-6) &= 0 \quad \Longrightarrow \quad x_1 = -2 \quad x_2 = 6
\end{aligned}$$

Thus, there are two $x-$intercepts, $(-2, 0)$ and $(6, 0)$. We will compute a few more points before graphing the parabola. We will work with $x-$coordinates close to that of the vertex, and use the standard form of the parabola, $y = -5(x-2)^2 + 80$.

$$\begin{aligned}
\text{if } x &= -3, \text{ then } y = -5\left(-3-2\right)^2 + 80 = -45 \quad \Longrightarrow \quad \text{found} \quad (-3,-45) \\
\text{if } x &= -1, \text{ then } y = -5\left(-1-2\right)^2 + 80 = 35 \quad \Longrightarrow \quad \text{found} \quad (-1,35) \\
\text{if } x &= 1, \text{ then } y = -5\left(1-2\right)^2 + 80 = 75 \quad \Longrightarrow \quad \text{found} \quad (1,75) \\
\text{if } x &= 3, \text{ then } y = -5\left(3-2\right)^2 + 80 = 75 \quad \Longrightarrow \quad \text{found} \quad (3,75) \\
\text{if } x &= 4, \text{ then } y = -5\left(4-2\right)^2 + 80 = 60 \quad \Longrightarrow \quad \text{found} \quad (4,60) \\
\text{if } x &= 5, \text{ then } y = -5\left(5-2\right)^2 + 80 = 35 \quad \Longrightarrow \quad \text{found} \quad (6,-5) \\
\text{if } x &= 7, \text{ then } y = -5\left(7-2\right)^2 + 80 = -45 \quad \Longrightarrow \quad \text{found} \quad (7,-45)
\end{aligned}$$

We are ready to graph: we have the following points, listed left to right:

	$(-3,-45)$
$x-$intercept	$(-2,0)$
	$(-1,35)$
$y-$intercept	$(0,60)$
	$(1,75)$
vertex	$(2,80)$
	$(3,75)$
	$(4,60)$
	$(5,35)$
$x-$intercept	$(6,0)$
	$(7,-45)$



Example 2. Graph the parabola $y = 4x + 2x^2 - 70$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution:

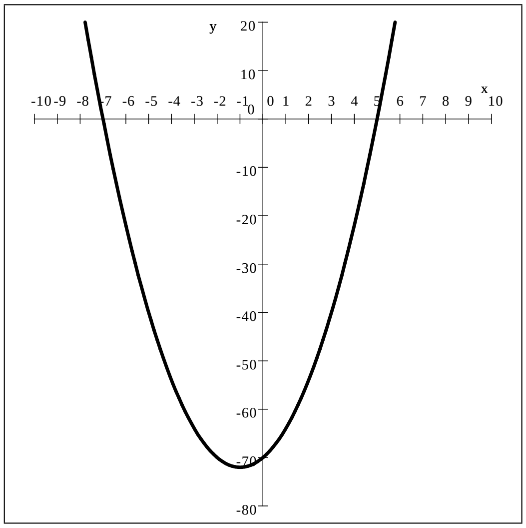
$$y = 2x^2 + 4x - 70 \qquad \textbf{polynomial form} \quad \Longrightarrow \quad y - \text{intercept: } (0,-70)$$

We factor out the leading coefficient

$$\begin{aligned}
y &= 2\left(x^2 + 2x - 35\right) \qquad\qquad (x+1)^2 = x^2 + 2x + 1 \\
y &= 2\left(\underbrace{x^2 + 2x + 1} - 1 - 35\right) \\
y &= 2\left((x+1)^2 - 36\right) = 2\left(x+1\right)^2 - 72 \qquad \textbf{standard form} \quad \Longrightarrow \quad \text{vertex: } (-1,-72) \\
y &= 2\left((x+1)^2 - 6^2\right) \\
y &= 2\left(x+1+6\right)\left(x+1-6\right) \\
y &= 2\left(x+7\right)\left(x-5\right) \qquad \textbf{factored form} \quad \Longrightarrow \quad x - \text{intercepts } (-7,0), \quad (5,0)
\end{aligned}$$

We find a few additional points, close to the vertex. We are ready to graph: we have the following points, listed left to right:

	$(-8, 26)$
$x-$intercept	$(-7, 0)$
	$(-5, -40)$
	$(-4, -54)$
	$(-3, -64)$
	$(-2, -70)$
vertex	$(-1, -72)$
$y-$intercept	$(0, -70)$
	$(1, -64)$
	$(2, -54)$
	$(3, -40)$
	$(4, -22)$
$x-$intercept	$(5, 0)$
	$(6, 26)$



Example 3. Graph the parabola $y = 8x - \frac{1}{2}x^2 - 32$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution: Since the leading coefficient is negative, the graph will be that of a downward opening parabola.

$$y = -\frac{1}{2}x^2 + 8x - 32 \qquad \textbf{polynomial form} \quad \Longrightarrow \quad y - \text{intercept: } (0, -32)$$

We factor out the leading coefficient, $-\frac{1}{2}$

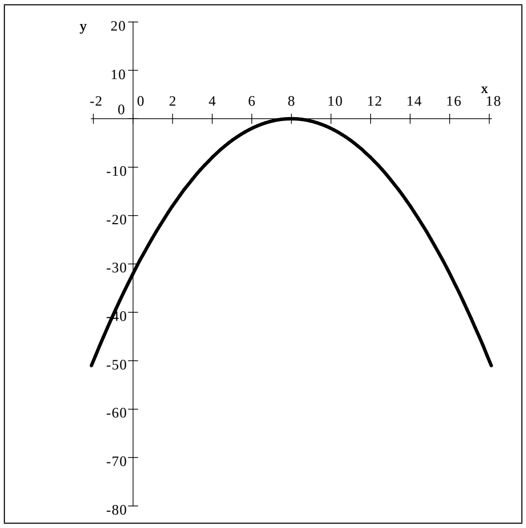
$$y = -\frac{1}{2}\left(x^2 - 16x + 64\right) \qquad (x-8)^2 = x^2 - 16x + 64$$

As we are getting ready to complete the square, we notice that the expression in the parentheses is a complete square. This means that $y = -\frac{1}{2}(x-8)^2$ is the **standard form** and the **factored form** all in one.

For the vertex, we can think of the equation as $y = -\frac{1}{2}(x-8)^2 + 0$ and so the vertex is $(8, 0)$. For the $x-$intercepts, we can think of the equation as $y = -\frac{1}{2}(x-8)(x-8)$ and so there is one $x-$intercept, $(8, 0)$ that is also the vertex.

We find a few additional points, close to the vertex and then graph the parabola.

$y-$intercept	$(0,-32)$
	$\left(1,-\frac{49}{2}\right)$
	$(2,-18)$
	$(4,-8)$
	$\left(5,-\frac{9}{2}\right)$
	$(6,-2)$
	$\left(7,-\frac{1}{2}\right)$
vertex and $x-$intercept	$(8,0)$
	$\left(9,-\frac{1}{2}\right)$
	$(10,-2)$
	$(12,-8)$
	$\left(13,-\frac{25}{2}\right)$



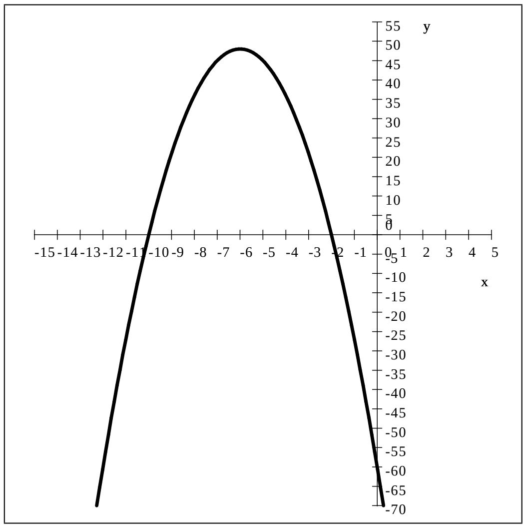
Example 4: Graph the parabola $y=-36x-3x^2-60$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution: Since the leading coefficient is negative, the graph will be a downward opening parabola.

$$
\begin{aligned}
y &= -3x^2-36x-60 && \Longrightarrow \quad y-\text{intercept: } (0,-60)\\
y &= -3\left(x^2+12x+20\right) && (x+6)^2 = x^2+12x+36\\
y &= -3\left(\underbrace{x^2+12x+36}-36+20\right)\\
y &= -3\left((x+6)^2-16\right) = -3(x+6)^2+48 && \Longrightarrow \quad \text{vertex: } (-6,48)\\
y &= -3\left((x+6)^2-4^2\right)\\
y &= -3(x+10)(x+2)\\
y &= -3(x+10)(x+2) && \Longrightarrow \quad x-\text{intercepts } (-10,0),\ (-2,0)
\end{aligned}
$$

We find a few more points close to the vertex and graph the parabola.

	$(-13,-99)$
	$(-12,-60)$
	$(-11,-27)$
$x-$intercept	$(-10,0)$
	$(-9,21)$
	$(-8,36)$
	$(-7,45)$
vertex	$(-6,48)$
	$(-5,45)$
	$(-4,36)$
	$(-3,21)$
$x-$intercept	$(-2,0)$
	$(-1,-27)$
$y-$intercept	$(0,-60)$



Example 5. Graph the parabola $y = 9 - x^2$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution: Since the leading coefficient is negative, the graph will be a downward opening parabola.

$$y = -x^2 + 9 \qquad \textbf{polynomial form} \quad \Longrightarrow \quad y - \text{intercept: } (0,9)$$

Because the linear term is missing, we do not need to complete the square, the polynomial form is also the standard form. If it helps, we can think of $-x^2 + 9$ as $-(x-0)^2 + 9$ to see that the vertex is $(0,9)$.

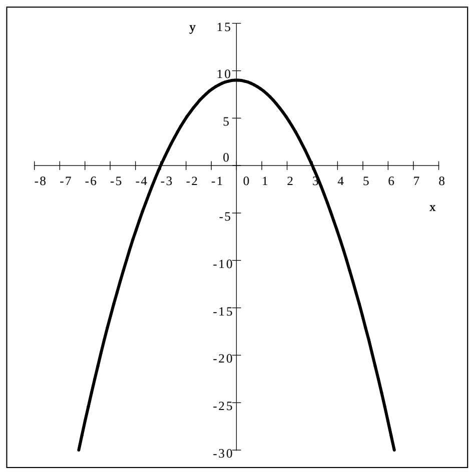
$$y = -x^2 + 9 \qquad \textbf{standard form} \quad \Longrightarrow \quad \text{vertex: } (0,9)$$

For the factored form, we factor out the leading coefficient and factor via the difference of squares theorem.

$$\begin{aligned} y &= -\left(x^2 - 9\right) = -\left(x^2 - 3^2\right) \\ y &= -(x+3)(x-3) \qquad \textbf{factored form} \quad \Longrightarrow \quad x - \text{intercepts } (-3,0), \ (3,0) \end{aligned}$$

We find a few additional points close to the vertex and then graph the parabola.

	$(-5,-16)$
	$(-4,-7)$
$x-$intercept	$(-3,0)$
	$(-2,5)$
	$(-1,8)$
vertex and $y-$intercept	$(0,9)$
	$(1,8)$
	$(2,5)$
$x-$intercept	$(3,0)$
	$(4,-7)$
	$(5,-16)$



Example 6. Graph the parabola $y = 6x^2 - 36x + 78$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution: Since the leading coefficient is positive, the graph will be an upward opening parabola. We first factor out the leading coefficient and then complete the square to obtain the standard form.

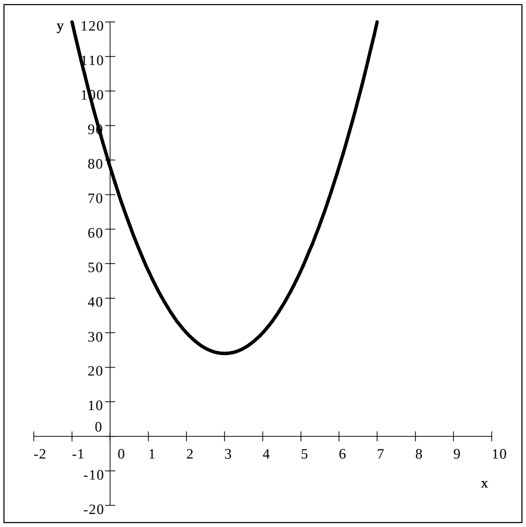
$$\begin{aligned} y &= 6x^2 - 36x + 78 && \Longrightarrow \quad y - \text{intercept: } (0,78) \\ y &= 6\left(x^2 - 6x + 13\right) && \qquad (x-3)^2 = x^2 - 6x + 9 \\ y &= 6\left(\underbrace{x^2 - 6x + 9} - 9 + 13\right) && \\ y &= 6\left((x-3)^2 + 4\right) = 6(x-3)^2 + 24 && \Longrightarrow \quad \text{vertex: } (3,24) \end{aligned}$$

This parabola has no $x-$intercepts. This is because it is an upward opening parabola with its vertex above the $x-$axis. Indeed, the expression $6(x-3)^2 + 24$ is always at least 24. Consequently, this expression is never zero.

Algebraically, this means that the factored form can not exist, because it would certainly guarantee $x-$intercepts. Indeed, this parabola does not have a factored form of its equation because the sum of squares in the parentheses in $6\left((x-3)^2 + 4\right)$ can not be factored. Recall that **the sum of two squares can never be factored.**

We find a few more points close to the vertex and graph the parabola.

	$(-1, 120)$
$y-$intercept	$(0, 78)$
	$(1, 48)$
	$(2, 30)$
vertex	$(3, 24)$
	$(4, 30)$
	$(5, 48)$
	$(6, 120)$



Example 7. We are standing on the top of a 720 ft tall building and throw a small object upward. The object's distance from the ground, measured in feet, after t seconds is

$$y = -16t^2 + 192t + 720$$

What is the highest point that the object reaches?

Solution: $y = -16t^2 + 192t + 720$ is quadratic, with a negative leading coefficient. Consequently, the graph is a downward opening parabola. The highest point is the vertex. We will find the vertex by completing the square.

$$\begin{aligned} y &= -16t^2 + 192t + 720 = -16\left(t^2 - 12t - 45\right) && (t-6)^2 = t^2 - 12t + 36 \\ y &= -16\left(t^2 - 12t + 36 - 36 - 45\right) \\ y &= -16\left((t-6)^2 - 81\right) && \text{distribute } -16 \\ y &= -16(t-6)^2 + 1296 \end{aligned}$$

This indicates that the parabola has its vertex at $(6, 1296)$. Thus the highest point is reached 6 seconds after we threw the object, and it is at a height of 1296 feet.

Example 8. A company finds that if they price their product at \$60, they can sell 500 items of it. For every dollar increase in the price, the number of items sold will decrease by 5.

a) Let x be the increase in price from \$60. Define the revenue function, $R(x)$ to be the sales revenue that results in such pricing. Find a formula for $R(x)$.

b) What price would guarantee an income of \$31 500?

c) Find the price that guarantees the maximum revenue.

d) Find the maximum revenue.

Solution: a) Let x be the increase in price from \$60. Define the revenue function, $R(x)$ to be the sales revenue that results in such pricing. Find a formula for $R(x)$.

Solution: Revenue = (Price) $\cdot$ (Number of items sold)

$$\begin{aligned} R(x) &= (60 + x)(500 - 5x) && \text{rearrange} \\ &= (x + 60)(-5x + 500) && \text{factor out } -5 \\ &= -5(x + 60)(x - 100) \\ &= -5\left(x^2 - 100x + 60x - 6000\right) \\ &= -5\left(x^2 - 40x - 6000\right) \end{aligned}$$

b) What price would guarantee an income of \$31 500?
Solution: We have to solve the equation: $x = ?$ so that $R(x) = 31500$.

$$x = ? \quad \text{so that} \quad R(x) = 31500$$

$$\text{Solve} \quad -5x^2 + 200x + 30000 = 31500 \quad \text{for } x$$

Since the equation is quadratic, we will reduce one side to zero. We will avoid a negative leading coefficient by reducing the left-hand side to zero.

$$\begin{aligned} -5x^2 + 200x + 30000 &= 31500 \\ 0 &= 5x^2 - 200x + 1500 \end{aligned}$$

We will factor by completing the square.

$$\begin{aligned} 5x^2 - 200x + 1500 &= 0 \quad \text{factor out } 5 \\ 5\left(x^2 - 40x + 300\right) &= 0 \qquad (x-20)^2 = x^2 - 40x + 400 \\ 5\left(\underbrace{x^2 - 40x + 400} - 400 + 300\right) &= 0 \\ 5\left((x-20)^2 - 100\right) &= 0 \\ 5\left((x-20)^2 - 10^2\right) &= 0 \\ 5(x - 20 + 10)(x - 20 - 10) &= 0 \\ 5(x-10)(x-30) &= 0 \implies x_1 = 10 \qquad x_2 = 30 \end{aligned}$$

We check. If $x = 10$, the price is increased by \$10 and so it sells for \$70. We can then sell $500 - 5(10) = 450$ items, and so the income is $R(10) = 70 \cdot 450 = 31\,500$. On the other hand, if $x = 30$, the selling price is \$ 90. We then can sell $500 - 5(30) = 350$ items, and so the revenue is $R(30) = 90 \cdot 350 = 31\,500$.
c) Find the price that guarantees the maximum revenue.
Solution: $R(x)$ is clearly a downward turning parabola. The maximum $y-$value is the y-value of the vertex. To find the vertex, we complete the square.

$$\begin{aligned} R(x) &= -5x^2 + 200x + 30000 \\ &= -5\left(x^2 - 40x - 6000\right) \qquad (x-20)^2 = x^2 - 40x + 400 \\ &= -5\left(\underbrace{x^2 - 40x + 400} - 400 - 6000\right) \\ &= -5\left((x-20)^2 - 6400\right) \qquad \text{distribute } -5 \\ &= -5(x-20)^2 + 32\,000 \end{aligned}$$

The vertex of the parabola is $(20, 32\,000)$. Thus a \$20 increase, implying a price of \$80, will guarantee a maximal income of \$32 000.
d) Find the maximum revenue. Solution: see part c).

Practice Problems

Graph each of the parabolas given below. In each case, clearly label the coordinates of five points of the parabola, including vertex and intercepts.

1. $y = 336 - 16x^2 - 64x$
2. $y = 3x^2 - 30x + 75$
3. $y = 4x - 2x^2 - 20$
4. $y = 6x - x^2$
5. $y = 3x^2 - 12$
6. $y = 4x + \frac{1}{2}x^2 + 10$
7. $y = x(2 - x) - 1$
8. $y = -12x - 2x^2 - 10$
9. We are standing on the top of a 720 ft tall building and throw a small object upward. The object's distance from the ground, measured in feet, after t seconds is

$$y = -16t^2 + 64t + 720$$

What is the highest point that the object reaches?

10. A company finds that if they price their product at \$20, they can sell 3000 items of it. For every dollar increase in the price, the number of items sold will decrease by 10. Find the price that guarantees the maximum revenue. Find the maximum revenue possible.

Practice Problems - Answers

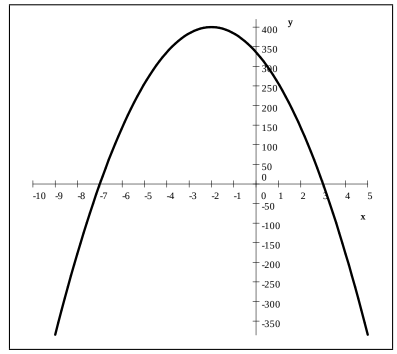
1.) $y = 336 - 16x^2 - 64x$

polynomial form: $y = -16x^2 - 64x + 336$ $y-$intercept: $(0, 336)$

standard form: $y = -16(x+2)^2 + 400$ vertex: $(-2, 400)$

factored form: $y = -16(x+7)(x-3)$
$x-$intercepts: $(-7, 0)$ and $(3, 0)$

additional points: $(-5, 256)$, $(-4, 336)$, $(-3, 384)$, $(-1, 384)$, $(1, 256)$, $(2, 144)$, $(4, -176)$



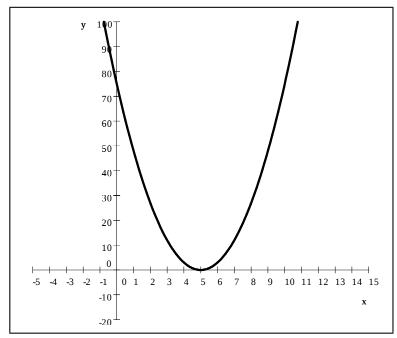
2.) $y = 3x^2 - 30x + 75$

polynomial form: $y = 3x^2 - 30x + 75$ $y-$intercept: $(0, 75)$

standard form: $y = 3(x-5)^2$ vertex: $(5, 0)$

factored form: $y = 3(x-5)^2$ $x-$intercept: $(5, 0)$

additional points: $(1, 48)$, $(2, 27)$, $(3, 12)$, $(4, 3)$, $(6, 3)$, $(7, 12)$, $(8, 27)$



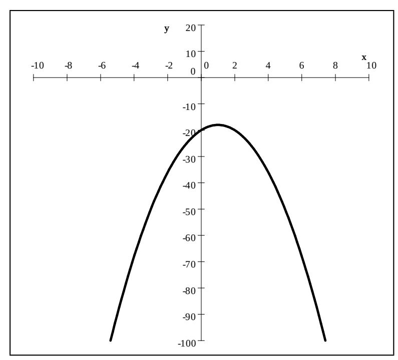
3.) $y = 4x - 2x^2 - 20$

polynomial form: $y = -2x^2 + 4x - 20$
$y-$intercept: $(0, -20)$

standard form: $y = -2(x-1)^2 - 18$ vertex: $(1, -18)$

factored form: doesn't exist, no $x-$intercepts

additional points: $(-3, -50)$, $(-2, -36)$, $(2, -20)$, $(3, -26)$, $(4, -36)$, $(5, -50)$, $(6, -68)$



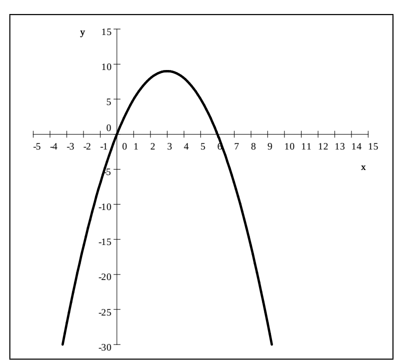
4.) $y = 6x - x^2$

polynomial form: $y = -x^2 + 6x$ $y-$intercept: $(0, 0)$

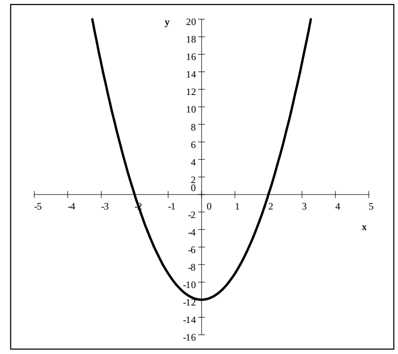
standard form: $y = -(x-3)^2 + 9$ vertex: $(3, 9)$

factored form: $y = -x(x-6)$ $x-$intercepts: $(0, 0)$ and $(6, 0)$

additional points: $(-1, -7)$, $(1, 5)$, $(2, 8)$, $(4, 8)$, $(5, 5)$, $(7, -7)$, $(8, -16)$



5.) $y = 3x^2 - 12$



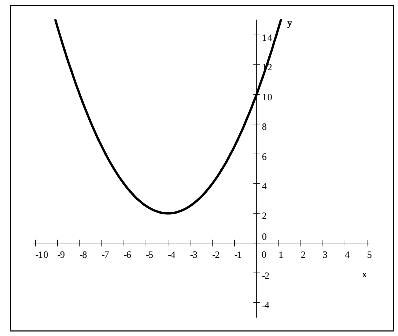
polynomial form: $y = 3x^2 - 12$ $\quad$ $y-$intercept: $(0, -12)$

standard form: $y = 3x^2 - 12$ $\quad$ vertex: $(0, -12)$

factored form: $y = 3(x+2)(x-2)$ $\quad$ $x-$intercepts: $(-2, 0)$ and $(2, 0)$

additional points: $(-4, 36)$, $(-3, 15)$, $(-1, -9)$, $(1, -9)$, $(3, 15)$, $(4, 36)$, $(5, 63)$

6.) $y = 4x + \frac{1}{2}x^2 + 10$



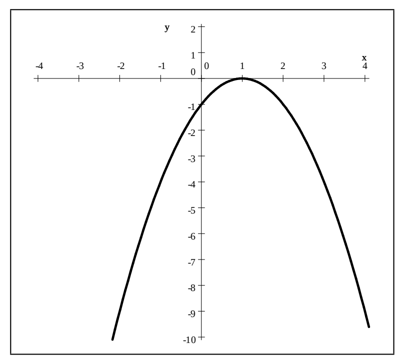
polynomial form: $y = \frac{1}{2}x^2 + 4x + 10$ $\quad$ $y-$intercept: $(0, 10)$

standard form: $y = \frac{1}{2}(x+4)^2 + 2$ $\quad$ vertex: $(-4, 2)$

factored form: doesn't exist, no $x-$intercepts

additional points: $\left(-7, \frac{13}{2}\right)$, $(-6, 4)$, $\left(-5, \frac{5}{2}\right)$, $\left(-3, \frac{5}{2}\right)$, $(-2, 4)$ $\left(-1, \frac{13}{2}\right)$, $\left(1, \frac{29}{2}\right)$, $(2, 20)$

7.) $y = x(2-x) - 1$



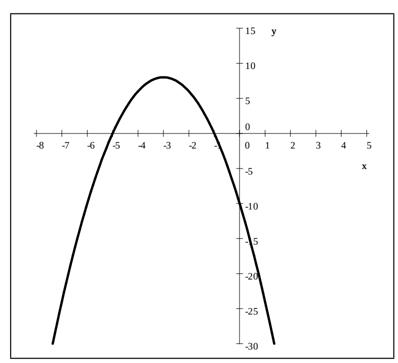
polynomial form: $y = -x^2 + 2x - 1$ $\quad$ $y-$intercept: $(0, -1)$

standard form: $y = -(x-1)^2$ $\quad$ vertex: $(1, 0)$

factored form: $y = -(x-1)^2$ $\quad$ $x-$intercept: $(1, 0)$

additional points: $(-2, -9)$, $(-1, -4)$, $(2, -1)$, $(3, -4)$, $(4, -9)$, $(5, -16)$

8.) $y = -12x - 2x^2 - 10$



polynomial form: $y = -2x^2 - 12x - 10$ $\quad$ $y-$intercept: $(0, -10)$

standard form: $y = -2(x+3)^2 + 8$ $\quad$ vertex: $(-3, 8)$

factored form: $y = -2(x+5)(x+1)$
$x-$intercepts: $(-5, 0)$ and $(-1, 0)$

additional points: $(-7, -24)$, $(-6, -10)$, $(-4, 6)$, $(-2, 6)$, $(1, -24)$, $(2, -42)$

9. 784 feet $\quad$ 10. max income: \$256 000 when the price is \$160

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