Part 1 - Deriving the Formula

Let $ax^2+bx+c=0$ be a quadratic equation, where $a\neq 0$. We will solve this equation by completing the square. We first factor out the leading coefficient, a.

$$\begin{aligned} ax^2+bx+c &= 0 \\ a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) &= 0 \end{aligned}$$

Half of the linear coefficient is $\dfrac{b}{2a}$ and so the complete square we need is

$$\left(x+\frac{b}{2a}\right)^2=\left(x+\frac{b}{2a}\right)\left(x+\frac{b}{2a}\right)=x^2+\frac{b}{2a}x+\frac{b}{2a}x+\frac{b^2}{4a^2}=x^2+\frac{b}{a}x+\frac{b^2}{4a^2}$$

Now we "smuggle in" the last missing term of the complete square, $\dfrac{b^2}{4a^2}$

$$\begin{aligned} a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) &= 0 \\ a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right) &= 0 \\ a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right) &= 0 \end{aligned}$$

We bring the fractions after the complete square to the common denominator, and factor out -1

$$\begin{aligned} a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}-\frac{-c\cdot 4a}{a\cdot 4a}\right) &= 0 \\ a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}-\frac{-4ac}{4a^2}\right) &= 0 \\ a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right) &= 0 \end{aligned}$$

We prepare to factor via the difference of squares theorem. If b^2-4ac is negative, then the expression $-\dfrac{b^2-4ac}{4a^2}$ is positive and the equation is

$$a\left(\left(x+\frac{b}{2a}\right)^2+p\right)=0 \quad \text{for some positive } p$$

Then this equation has no real solution.
If b^2-4ac is positive or zero, then we can factor via the difference of squares theorem

$$a\left(\left(x+\frac{b}{2a}\right)^2-\left(\sqrt{\frac{b^2-4ac}{4a^2}}\right)^2\right)=0$$

The second expression can be simplified since

$$\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}=\frac{\sqrt{b^2-4ac}}{|2a|}$$

We will further simplfy this and write $2a$ instead of $|2a|$. We factor via the difference of squares theorem

$$a\left(\left(x+\frac{b}{2a}\right)^2-\left(\sqrt{\frac{b^2-4ac}{4a^2}}\right)^2\right) = 0$$

$$a\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right) = 0$$

We apply the zero product rule and solve for x in each of the two equations.

$$\begin{aligned} x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a} &= 0 \\ x+\frac{\sqrt{b^2-4ac}}{2a} &= -\frac{b}{2a} \\ x &= -\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a} \\ x &= \frac{-b-\sqrt{b^2-4ac}}{2a} \end{aligned}$$

or

$$\begin{aligned} x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a} &= 0 \\ x-\frac{\sqrt{b^2-4ac}}{2a} &= -\frac{b}{2a} \\ x &= -\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a} \\ x &= \frac{-b+\sqrt{b^2-4ac}}{2a} \end{aligned}$$

Theorem: (The quadratic formula) Suppose that a, b, and c are real numbers, where $a \neq 0$. Consider the quadratic equation

$$ax^2+bx+c=0$$

If b^2-4ac is negative, the equation has no real solution. If b^2-4ac is non-negative, than the solution(s) of the quadratic equation are x_1 and x_2 and they can be computed by the formula

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The expression b^2-4ac is called the **discriminant**. When the discriminant is positive, the equation has two different real solutions. When the discriminant is zero, the equation has exactly one real solution. When the discriminant is negative, the equation has no real solution.

Example 1. Solve $2x^2-9x=5$ over the real numbers.

Solution: We first reduce one side to zero: $2x^2-9x-5=0$. Now we can apply the quadratic formula.
$a=2$, $b=-9$, and $c=-5$

$$x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-9)\pm\sqrt{(-9)^2-4\cdot 2(-5)}}{2\cdot 2}=\frac{9\pm\sqrt{81-(-40)}}{4}=\frac{9\pm\sqrt{81+40}}{4}$$

$$= \frac{9\pm\sqrt{121}}{4}=\frac{9\pm 11}{4}=\begin{cases}\dfrac{9+11}{4}=\dfrac{20}{4}=5\\[2ex] \dfrac{9-11}{4}=\dfrac{-2}{4}=-\dfrac{1}{2}\end{cases}$$

So this equation has two real solutions, $\boxed{5 \text{ and } -\frac{1}{2}}$. We check: if $x=5$, then

$$\text{LHS}=2(5)^2-9(5)=50-45=5=\text{RHS}$$

and if $x=-\frac{1}{2}$, then $\text{LHS}=2\left(-\frac{1}{2}\right)^2-9\left(-\frac{1}{2}\right)=2\cdot\frac{1}{4}+\frac{9}{2}=\frac{1}{2}+\frac{9}{2}=\frac{10}{2}=5=\text{RHS}$ and so both solutions are correct.

Example 2. Solve $x^2 - 7x = -10 - x$ over the real numbers.

Solution: We first reduce one side to zero.

$$\begin{aligned} x^2 - 7x &= -10 - x && \text{add } x \\ x^2 - 6x &= -10 && \text{add } 10 \\ x^2 - 6x + 10 &= 0 \end{aligned}$$

Now we can apply the quadratic formula with values $a = 1$, $b = -6$, and $c = 10$.

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1} = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm \sqrt{-4}}{2} = \text{undefined}$$

We now see that the discriminant is negative. This means that the equation has $\boxed{\text{no real solutions}}$.

Example 3. Solve $2x^2 - 10x + 26 = 6x$ over the real numbers.

Solution: Notice that each term is even, and so we can divide both sides by 2.

$$\begin{aligned} 2x^2 - 10x + 26 &= 6x && \text{subtract } 6x \\ 2x^2 - 16x + 26 &= 0 \\ 2\left(x^2 - 8x + 13\right) &= 0 \\ x^2 - 8x + 13 &= 0 \end{aligned}$$

Now we can apply the quadratic formula with values $a = 1$, $b = -8$, and $c = 13$.

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1} = \frac{8 \pm \sqrt{64 - 52}}{2} = \frac{8 \pm \sqrt{12}}{2}$$

However, this is not the most simplified form of the solution. Recall that $\sqrt{ab} = \sqrt{a}\sqrt{b}$ for all non-negative real numbers a and b. Using this rule, we obtain that $\sqrt{12} = 2\sqrt{3}$ as follows.

$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$$

The solution can be further simplified:

$$x_{1,2} = \frac{8 \pm \sqrt{12}}{2} = \frac{8 \pm 2\sqrt{3}}{2} = \frac{2\left(4 \pm \sqrt{3}\right)}{2} = 4 \pm \sqrt{3}$$

This of course means that $\boxed{x_1 = 4 + \sqrt{3} \text{ and } x_2 = 4 - \sqrt{3}}$.
We check: if $x = 4 + \sqrt{3}$, then

$$\begin{aligned} \text{LHS} &= 2\left(4 + \sqrt{3}\right)^2 - 10\left(4 + \sqrt{3}\right) + 26 = 2\left(16 + 3 + 8\sqrt{3}\right) - 10\left(4 + \sqrt{3}\right) + 26 \\ &= 2\left(9 + 8\sqrt{3}\right) - 10\left(4 + \sqrt{3}\right) + 26 = 18 + 16\sqrt{3} - 40 - 10\sqrt{3} + 26 = 24 + 6\sqrt{3} \\ \text{RHS} &= 6\left(4 + \sqrt{3}\right) = 24 + 6\sqrt{3} \end{aligned}$$

and so $x_1 = 4 + \sqrt{3}$ works. And if $x = 4 - \sqrt{3}$, then

$$\begin{aligned} \text{LHS} &= 2\left(4 - \sqrt{3}\right)^2 - 10\left(4 - \sqrt{3}\right) + 26 = 2\left(16 + 3 - 8\sqrt{3}\right) - 10\left(4 - \sqrt{3}\right) + 26 \\ &= 2\left(9 - 8\sqrt{3}\right) - 10\left(4 - \sqrt{3}\right) + 26 = 18 - 16\sqrt{3} - 40 + 10\sqrt{3} + 26 = 24 - 6\sqrt{3} \\ \text{RHS} &= 6\left(4 - \sqrt{3}\right) = 24 - 6\sqrt{3} \end{aligned}$$

and so $x_2 = 4 - \sqrt{3}$ also works.

Example 4. Solve $9x^2 + 1 = 6x$ over the real numbers.

Solution:

$$\begin{aligned} 9x^2 + 1 &= 6x \\ 9x^2 - 6x + 1 &= 0 \end{aligned}$$

We apply the quadratic formula with values $a = 9$, $b = -6$, and $c = 1$.

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 9 \cdot 1}}{2 \cdot 9} = \frac{6 \pm \sqrt{36 - 36}}{18} = \frac{6 \pm \sqrt{0}}{18} = \frac{6}{18} = \frac{1}{3}$$

So this equation has only one real solution: $x = \frac{1}{3}$. We check: if $x = \frac{1}{3}$, then

$$\text{LHS} = 9\left(\frac{1}{3}\right)^2 + 1 = 9 \cdot \frac{1}{9} + 1 = 2 \text{ and RHS} = 6 \cdot \frac{1}{3} = \frac{6}{3} = 2$$

And so our solution is correct.

Example 5. In each of the given equations, use the discriminant to determine the number of real solutions without solving the equation.

a) $5x^2 - 3x + 1 = 0$

b) $-2x^2 + 7x + 2 = 0$

c) $\frac{1}{2}x^2 - x + 1 = 0$

d) $8x^2 + \frac{1}{2} = 4x$

e) $3x^2 - 12x + 87 = 0$

f) $13x + 3x^2 = 56$

g) $9x^2 - 30x + 25 = 0$

Solution: We will compute the value of the discriminant, $D = b^2 - 4ac$. If the discriminant is positive, there are two positive solutions. If it is zero, there is exactly one real solution. If it is negative, there is no real solution.

a) $5x^2 - 3x + 1 = 0$. In this case, $a = 5$, $b = -3$, and $c = 1$.

$$D = b^2 - 4ac = (-3)^2 - 4 \cdot 5 \cdot 1 = 9 - 20 = -11$$

Since the discriminant is negative, there is no real solution for this equation.

b) $-2x^2 + 7x + 2 = 0$. In this case, $a = -2$, $b = 7$, and $c = 2$.

$$D = b^2 - 4ac = 7^2 - 4(-2)2 = 49 - (-16) = 65$$

Since the discriminant is positive, there are two real solutions for this equation.

c) $\frac{1}{2}x^2 - x + 1 = 0$. In this case, $a = \frac{1}{2}$, $b = -1$, and $c = 1$.

$$D = b^2 - 4ac = \left(\frac{1}{2}\right)^2 - 4 \cdot \frac{1}{2} \cdot 1 = \frac{1}{4} - 2 = \frac{1}{4} - \frac{8}{4} = -\frac{7}{4}$$

Since the discriminant is negative, there is no real solution for this equation.

d) $8x^2 + \frac{1}{2} = 4x$.

Solution: Caution! We can not apply the quadratic formula if the equation is in this form. Remember, one side must be zero.

$$\begin{aligned} 8x^2 + \frac{1}{2} &= 4x \qquad \text{subtract } 4x \\ 8x^2 - 4x + \frac{1}{2} &= 0 \end{aligned}$$

And so in this case, $a = 8$, $b = -4$, and $c = \frac{1}{2}$.

$$D = b^2 - 4ac = (-4)^2 - 4 \cdot 8 \cdot \frac{1}{2} = 16 - 16 = 0$$

Since the discriminant is zero, there is exactly one real solution for this equation.

e) $3x^2 - 12x + 87 = 0$

Solution: At first it appears that $a = 3$, $b = -12$, and $c = 87$. We might notice that all of a, b, and c are divisible by 3. It might make things easier if we divide both sides of the equation by 3 before applying the quadratic formula.

$$\begin{aligned} 3x^2 - 12x + 87 &= 0 \\ 3\left(x^2 - 4x + 29\right) &= 0 \qquad \text{divide by 3} \\ x^2 - 4x + 29 &= 0 \end{aligned}$$

So now $a = 1$, $b = -4$, and $c = 29$.

$$D = b^2 - 4ac = (-4)^2 - 4 \cdot 1 \cdot 29 = 16 - 116 = -100$$

Since the discriminant is negative, there is no real solution for this equation.

f) $13x + 3x^2 = 56$

Solution: We first reduce one side to zero. $3x^2 + 13x - 56 = 0$. So $a = 3$, $b = 13$, and $c = -56$.

$$D = b^2 - 4ac = 13^2 - 4 \cdot 3(-56) = 169 - (-672) = 169 + 672 = 841$$

Since the discriminant is positive, there are two real solutions for this equation.

g) $9x^2 - 30x + 25 = 0$

Solution: $a = 9$, $b = -30$, and $c = 25$.

$$D = b^2 - 4ac = 30^2 - 4 \cdot 9 \cdot 25 = 900 - 900 = 0$$

Since the discriminant is zero, there is exactly one real solution for this equation.

Practice Problems

1. Use the discriminant to determine how many real solutions are there for each of the given equation. You do not have to find the solutions.

   a) $-3x^2 + 2x - 8 = 0$
   b) $5x^2 - 1 = 2x$
   c) $0.1x^2 + x + 1 = 0$
   d) $\frac{4}{9}x^2 - \frac{4}{3}x + 1 = 0$
   e) $x^2 - 6x = -7$
   f) $x^2 - 8x + 20 = 0$
   g) $-2x^2 - 5x + 10 = 0$

2. Use the quadratic formula to solve each of the following equations over the real numbers.

   a) $6x^2 - x + 9 = 0$
   b) $6x + 2x^2 + 1 = x + 4$
   c) $9x^2 - 12x = 1$
   d) $x + 40x^2 = 15$
   e) $-2x^2 + 5x - 4 = 0$
   f) $x^2 + x = 1$
   g) $\frac{16}{25}x^2 - \frac{16}{5}x + 4 = 0$

3. Find all positive numbers with the property that it is one greater than its own reciprocal.

Answers - Practice Problems

1. a) no real solutions b) two real solutions c) two real solutions d) one real solution
   e) two real solutions f) no real solutions g) two real solutions

2. a) no real solutions b) $-3, \dfrac{1}{2}$ c) $\dfrac{2 \pm \sqrt{5}}{3}$ d) $-\dfrac{5}{8}, \dfrac{3}{5}$ e) no real solutions f) $\dfrac{-1 \pm \sqrt{5}}{2}$ g) $\dfrac{5}{2}$

3. $\dfrac{1 + \sqrt{5}}{2}$ (the other solution is negative)

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