We will now study radical equations. As the name suggests, these are equations with square roots or cubic roots, etc. Let us first recall a few facts.

Example 1. Simplify each of the given expressions.

a) $\left(\sqrt{x-2}\right)^2$ b) $\left(\sqrt{x}-2\right)^2$ c) $\left(-3\sqrt{x}\right)^2$

Solution: a) Recall that $\sqrt{5}$ is the non-negative number whose square is 5. Similarly, $\sqrt{x-2}$ is the non-negative quantity that, when squared, the result is $x-2$. That is exactly what happens here and so

$\left(\sqrt{x-2}\right)^2 = \boxed{x-2}$

b) This is a different situation. $\sqrt{x}$ is the non-negative number, that, when squared, the result is x. To square $\sqrt{x}-2$, we need to apply the distributive property.

$\left(\sqrt{x}-2\right)^2 = \left(\sqrt{x}-2\right)\left(\sqrt{x}-2\right) = \sqrt{x}\sqrt{x} - 2\sqrt{x} - 2\sqrt{x} + 4 = \left(\sqrt{x}\right)^2 - 4\sqrt{x} + 4 = \boxed{x - 4\sqrt{x} + 4}$

This computation shows that the radical is not always eliminated just becuse we square the expression.

b) $\left(-3\sqrt{x}\right)^2 = \left(-3\sqrt{x}\right)\left(-3\sqrt{x}\right) = 9\left(\sqrt{x}\right)^2 = \boxed{9x}$

When we are dealing with radical expressions and want to get rifd of radicals, squaring is useful for expressions such as $\sqrt{x-2}$ or $-3\sqrt{x}$, but not for expressions such as $\sqrt{x}-2$. This will be important to keep in mind.

Also recall that when solving an equations, an **equivalent step** is one that does not change the solution set.

Example 2. Find all real solutions of the equation $\sqrt{x-1} = -x+3$

Solution: If we square both sides of the equation, the radical on the left will disappear, and the expression on the right becomes quadratic, resulting in a quadratic equation. So it looks like it is a good idea to square both sides first and then solve the quadratic equation we obtained.

$$\begin{array}{rcll} \sqrt{x-1} &=& -x+3 & \text{square both sides} \\ x-1 &=& (-x+3)^2 & \text{expand complete square} \\ x-1 &=& x^2-6x+9 & \text{subtract } x \\ -1 &=& x^2-7x+9 & \text{add 1} \\ 0 &=& x^2-7x+10 & \\ 0 &=& (x-2)(x-5) & \end{array}$$

Therefore, it appears that this eqaution has two solutions, 2 and 5. Let us check.

If $x=2$, then LHS $= \sqrt{2-1} = \sqrt{1} = 1$ and RHS $= -2+3 = 1$ and so $x=2$ works.

If $x=5$, then LHS $= \sqrt{5-1} = \sqrt{4} = 2$ and RHS $= -5+3 = -2$, and $2 \neq -2$, so $x=5$ is not a solution of theis equation. Thus thiis equation has one solution, $\boxed{x=2}$.

Until now, checking a solution was just a matter of making certain that we did not make a mistake. This is a different situation: our computation was correct, and yet we have a number that is a solution of the last equation, but not of the first. This is because squaring both sides of an equation is a **non-equaivalent step** that increases the solution set.

When we substituted $x = 5$ into the original equation, we had $2 = -2$, which is a false statement. But next we squared both sides, and the false statement $2 = -2$ became $4 = 4$, which is true. This is how $x = 5$ works after we squared both sides, but not before.

If we square both sides of an equation $L = R$, and solve the equation $L^2 = R^2$, then some of the solutions of $L^2 = R^2$ might be numbers for which $L = -R$. Such a number is called an **extreneous solution**. When we square both sides of an equation, we *must* check our solution(s) because squaring both sides of an equation is a non-equivalent step. In our previous example, $x = 5$ was an extreneous solution.

Example 3. Find all real solutions of the equation $11 = \sqrt{4x+1} + x$

Solution: If we squared both sides as they are, we would not eliminate the radical. Recall that sums involving radicals do not respond well to squaring both sides. Before we square, we need to isolate the radical expression on one side. Therefore, we will start by sutracting x.

$$\begin{aligned}
11 &= \sqrt{4x+1} + x && \text{subtract } x \\
11 - x &= \sqrt{4x+1} && \text{square both sides} \\
(11-x)^2 &= 4x+1 && \\
x^2 - 22x + 121 &= 4x + 1 && \text{subtract } x\text{, subtract } 1 \\
x^2 - 26x + 120 &= 0 && \\
(x-6)(x-20) &= 0 \quad \Longrightarrow \quad x_1 = 6, \quad x_2 = 20 &&
\end{aligned}$$

We check: If $x = 6$, then RHS $= \sqrt{4 \cdot 6 + 1} + 6 = \sqrt{25} + 6 = 5 + 6 = 11 =$ LHS ✓
If $x = 20$, then RHS $= \sqrt{4 \cdot 20 + 1} + 20 = \sqrt{81} + 20 = 9 + 20 = 29 \neq 11$. Thus $x = 20$ is not a solution. The only solution is $\boxed{x = 6}$.

Why would $x = 20$ show up as a solution? Let us substitute 20 into the equation we squared. That is the second line, $11 - x = \sqrt{4x+1}$. If we substitute 20 into x, we get $-9 = 9$. That is a false statement, but we see that it will become true after squaring both sides.

We have only seen equations so far that had one solution and one extreneous solution. This is not the only possibility: some radical equations have two solution, some have none.

Example 4. Find all real solutions of the equation $\sqrt{3x+1} - \sqrt{x-1} = 2$

Solution: If we squared both sides as they are, we would be left with expressions such as $2\sqrt{3x+1}\sqrt{x-1}$. To avoid that,we should isolate the radical expression. In this equation, however, there are two radical expressions, and we cannot isolate both. We should select the more complicated one, isolate that and square. Then we will only left with one radical expression, so we repeat the process of isolating it and then sqjaring both sides.

$$\begin{aligned}
\sqrt{3x+1} - \sqrt{x-1} &= 2 && \text{add } \sqrt{x-1} \\
\sqrt{3x+1} &= \sqrt{x-1} + 2 && \text{square} \\
3x + 1 &= \left(\sqrt{x-1} + 2\right)^2 &&
\end{aligned}$$

To expand $\left(\sqrt{x-1} + 2\right)^2$, we apply the distributive law:

$$\begin{aligned}
\left(\sqrt{x-1} + 2\right)^2 &= \left(\sqrt{x-1} + 2\right)\left(\sqrt{x-1} + 2\right) = \sqrt{x-1}\sqrt{x-1} + 2\sqrt{x-1} + 2\sqrt{x-1} + 4 \\
&= \left(\sqrt{x-1}\right)^2 + 4\sqrt{x-1} + 4 = x - 1 + 4\sqrt{x-1} + 4 = x + 4\sqrt{x-1} + 3
\end{aligned}$$

and so our equation is

$$\begin{aligned} 3x+1 &= \left(\sqrt{x-1}+2\right)^2 \\ 3x+1 &= x+4\sqrt{x-1}+3 && \text{subtract } x \\ 2x+1 &= 4\sqrt{x-1}+3 && \text{subtract } 3 \\ 2x-2 &= 4\sqrt{x-1} \end{aligned}$$

Notice that all coefficients are even. Therefore, we may dividde both sides by 2. This is a step that is not necessary, but it saves us work as the numbers don't get as large.

$$\begin{aligned} 2x-2 &= 4\sqrt{x-1} && \text{divide by } 2 \\ x-1 &= 2\sqrt{x-1} && \text{square both sides} \\ (x-1)^2 &= \left(2\sqrt{x-1}\right)^2 \\ x^2-2x+1 &= 4(x-1) \\ x^2-2x+1 &= 4x-4 && \text{subtract } 4x \\ x^2-6x+1 &= -4 && \text{add } 4 \\ x^2-6x+5 &= 0 \\ (x-1)(x-5) &= 0 \qquad \Longrightarrow \qquad x_1=1,\ x_2=5 \end{aligned}$$

We check both candidates. If $x=1$, then

$$\text{LHS} = \sqrt{3\cdot 1+1}-\sqrt{1-1} = \sqrt{4}-\sqrt{0} = 2-0 = 2 = \text{RHS} \checkmark$$

and if $x=5$, then

$$\text{LHS} = \sqrt{3\cdot 5+1}-\sqrt{5-1} = \sqrt{16}-\sqrt{4} = 4-2 = 2 = \text{RHS} \checkmark$$

In case of this equation, both $\boxed{1 \text{ and } 5}$ are solutions.

Enrichment

1. Consider the equation $x^2-x-10=8+\sqrt{x^2-x-16}$

   The problem here is that if we isolate the radical expression an square, we will end up with an equation of degree 4. It is worth a try, as we can solve some degree 4 equations, but chances are that this one would be tougher. Here is another method.

   Let us introduce a new variable, $a=\sqrt{x^2-x-16}$. Then x^2-x-10 on the left-hand side can be written as $x^2-x-10=x^2-x-16+6=a^2+6$.

   Substituting a on both sides, our equation becomes $a^2+6=8+a$. Solve for a. Once you have a, solve for x.

Sample Problems

1. $\sqrt{3x-2}=x$
2. $\sqrt{3x+4}+2=x$
3. $10+\sqrt{4x-7}=7$
4. $5+\sqrt{x+15}=x$
5. $2\sqrt{x-1}=x-4$
6. $2\sqrt{x+4}=1+\sqrt{2x+9}$
7. $5\sqrt{x}+1=3\sqrt{x}+17$
8. $\sqrt{2x+5}+5=x$
9. $\sqrt{2x+5}-\sqrt{x-1}=\sqrt{x+2}$
10. $\sqrt{10x-1}+2=-x$
11. $\sqrt{3x+1}-\sqrt{x-4}=3$
12. $\sqrt{x+10}+10=x$
13. $\sqrt{4x-11}=\sqrt{x-1}+\sqrt{x-4}$
14. $\sqrt{4x+6}=\sqrt{x+1}-\sqrt{x+5}$

Practice Problems

1. $\sqrt{3x-5}=4$
2. $2\sqrt{a-1}+7=1$
3. $3\sqrt{7x+1}+2=20$
4. $\sqrt{x+3}=x-9$
5. $2\sqrt{x+5}=x-3$
6. $\sqrt{2p+4}-\sqrt{p+3}=1$
7. $\sqrt[3]{x^3+16}=x+4$
8. $\sqrt{18+x}=x-2$
9. $\sqrt{w}+\sqrt{w+3}=3$
10. $\sqrt{2x+1}+\sqrt{5-x}=4$
11. $2\sqrt{x}-\sqrt{x-3}=\sqrt{x+7}$
12. $2\sqrt{y+4}+\sqrt{y-5}=\sqrt{9y+7}$
13. $\sqrt{b-2}+b=8$
14. $\sqrt{3x+1}-\sqrt{x+4}=1$
15. $\sqrt[3]{x-8}-\sqrt[3]{4x+1}=0$
16. $\sqrt{5m-9}=\sqrt{5m}-3$

17 $\sqrt{3k-5}-\sqrt{3k}=-1$

18. $\sqrt{1-x}+1=x+12$
19. $2=\sqrt{x^2+1}-x$
20. $\sqrt{3x+1}-4=8-2\sqrt{3x+1}$
21. $\sqrt{x}=\sqrt{10+3\sqrt{x}}$
22. $\sqrt[3]{x^3+7}=x+1$
23. $\sqrt{x-6}=\sqrt{x+2}-4$
24. $\sqrt{6x+7}-\sqrt{3x+3}=1$

Answers

Sample Problems

1. $1, 2$ 2. 7 3. no real solution 4. 10 5. 10 6. 0 7. 64 8. 10

9. 2 10. no solution 11. $5, 8$ 12. 15 13. 5 14. no real solution

Practice Problems

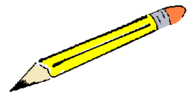
1. 7 2. no real solution 3. 5 4. 13 5. 11 6. 6 7. -2 8. 7 9. 1

10. $4, \frac{20}{9}$ 11. $\frac{25}{8}$ 12. 21 13. 6 14. 5 15. -3 16. $\frac{9}{5}$ 17. 3

18. -8 19. $-\frac{3}{4}$ 20. 5 21. 25 22. $-2, 1$ 23. no real solution 24. $\frac{1}{3}, -1$

Enrichment

1. $-4, 5$

Sample Problems  Solutions

1. $\sqrt{3x-2} = x$

   Solution:

$$\begin{aligned} \sqrt{3x-2} &= x && \text{square} \\ 3x-2 &= x^2 && \text{reduce one side to zero} \\ 0 &= x^2-3x+2 && \text{factor} \\ 0 &= (x-2)(x-1) && \Longrightarrow\ x_1 = 2 \ \text{ and } \ x_2 = 1 \end{aligned}$$

   We check: if $x = 2$, then $\quad \text{LHS} = \sqrt{3 \cdot 2 - 2} = \sqrt{4} = 2 = \text{RHS} \ \checkmark$

   and if $x = 1$, then $\quad \text{LHS} = \sqrt{3 \cdot 1 - 2} = \sqrt{1} = 1 = \text{RHS} \ \checkmark$

   Thus the solutions are $\boxed{1 \text{ and } 2}$

2. $\sqrt{3x+4} + 2 = x$

   Solution:

$$\begin{aligned} \sqrt{3x+4}+2 &= x && \text{subtract 2} \\ \sqrt{3x+4} &= x-2 && \text{square} \\ 3x+4 &= (x-2)^2 && \text{divide by 3} \\ 3x+4 &= x^2-4x+4 && \text{subtract } 3x \\ 4 &= x^2-7x+4 && \text{subtract 4} \\ 0 &= x^2-7x \\ 0 &= x(x-7) && \Longrightarrow\ x_1 = 2,\ x_2 = 1 \end{aligned}$$

   We check both candidates: if $x = 0$, then

$$\text{LHS} = \sqrt{3 \cdot 0 + 4} + 2 = \sqrt{4} + 2 = 2 + 2 = 4 \quad \text{and} \quad \text{RHS} = 2 \qquad \text{LHS} \neq \text{RHS}$$

   so 0 is not a solution. If $x = 7$, then

$$\text{LHS} = \sqrt{3 \cdot 7 + 4} + 2 = \sqrt{25} + 2 = 5 + 2 = 7 = \text{RHS} \ \checkmark$$

   Therefore, $\boxed{7}$ is the only solution.

3. $10 + \sqrt{4x-7} = 7$

   Solution:

$$\begin{aligned} 10 + \sqrt{4x-7} &= 7 && \text{subtract 10} \\ \sqrt{4x-7} &= -3 \end{aligned}$$

   Since the square root of no real number is negative, we can already see there is $\boxed{\text{no real solution}}$.

4. $5+\sqrt{x+15}=x$

Solution:

$$\begin{array}{rcll} 5+\sqrt{x+15} & = & x & \text{subtract 5} \\ \sqrt{x+15} & = & x-5 & \text{square both sides} \\ x+15 & = & (x-5)^2 & \text{FOIL right hand side} \\ x+15 & = & x^2-10x+25 & \text{reduce one side to zero} \\ 0 & = & x^2-11x+10 & \text{factor} \\ 0 & = & (x-1)(x-10) & \Longrightarrow\ x_1=1 \ \text{ and } x_2=10 \end{array}$$

We check: If $x=1$, then

$$\text{LHS}=5+\sqrt{1+15}=5+\sqrt{16}=5+4=9 \quad \text{and} \quad \text{RHS}=1 \qquad \text{RHS} \neq \text{LHS}$$

Thus $x=1$ is NOT a solution. If $x=10$, then

$$\text{LHS}=5+\sqrt{10+15}=5+\sqrt{25}=5+5=10=\text{RHS}\ \checkmark$$

Thus $x=\boxed{10}$ is the only solution.

5. $2\sqrt{x-1}=x-4$

Solution:

$$\begin{array}{rcll} 2\sqrt{x-1} & = & x-4 & \text{square both sides} \\ 4(x-1) & = & (x-4)^2 & \text{FOIL, distribute} \\ 4x-4 & = & x^2-8x+16 & \text{reduce one side to zero} \\ 0 & = & x^2-12x+20 & \text{factor} \\ 0 & = & (x-2)(x-10) & \Longrightarrow\ x_1=2 \ \text{ and } x_2=10 \end{array}$$

We check: If $x=2$, then

$$\text{LHS}=2\sqrt{2-1}=2\sqrt{1}=2\cdot 1=2 \ \text{ and } \ \text{RHS}=2-4=-2 \qquad \text{RHS} \neq \text{LHS}$$

Thus $x=2$ is NOT a solution. If $x=10$, then

$$\text{LHS}=2\sqrt{10-1}=2\sqrt{9}=2(3)=6=\text{RHS}\ \checkmark$$

Thus $\boxed{x=10}$ is the only solution.

6. $2\sqrt{x+4} = 1 + \sqrt{2x+9}$

Solution:

$$\begin{aligned}
2\sqrt{x+4} &= 1+\sqrt{2x+9} && \text{square both sides}\\
\left(2\sqrt{x+4}\right)^2 &= \left(1+\sqrt{2x+9}\right)^2 \\
2^2\left(\sqrt{x+4}\right)^2 &= \left(1+\sqrt{2x+9}\right)\left(1+\sqrt{2x+9}\right) \\
4(x+4) &= 1+\sqrt{2x+9}+\sqrt{2x+9}+2x+9 && \text{combine like terms}\\
4x+16 &= 2x+10+2\sqrt{2x+9} && \text{subtract } 2x\\
2x+16 &= 10+2\sqrt{2x+9} && \text{subtract } 10\\
2x+6 &= 2\sqrt{2x+9} && \text{factor out } 2
\end{aligned}$$

$$\begin{aligned}
2(x+3) &= 2\sqrt{2x+9} && \text{divide by } 2\\
x+3 &= \sqrt{2x+9} && \text{square both sides}\\
(x+3)^2 &= 2x+9 \\
x^2+6x+9 &= 2x+9 && \text{reduce one side to zero}\\
x^2+4x &= 0 && \text{factor}\\
x(x+4) &= 0 \implies x_1 = 0 \quad \text{and} \quad x_2 = -4
\end{aligned}$$

We check: If $x = 0$, then

$$\text{LHS} = 2\sqrt{0+4} = 2\sqrt{4} = 2\cdot 2 = 4 \text{ and } \text{RHS} = 1+\sqrt{2(0)+9} = 1+3 = 4 \checkmark$$

Thus $x = 0$ is indeed a solution. If $x = -4$, then

$$\begin{aligned}
\text{LHS} &= 2\sqrt{(-4)+4} = 2\sqrt{0} = 2(0) = 0\\
\text{RHS} &= 1+\sqrt{2(-4)+9} = 1+\sqrt{-8+9} = 1+\sqrt{1} = 1+1 = 2 \qquad \text{RHS} \neq \text{LHS}
\end{aligned}$$

Thus $x = -4$ is NOT a solution. The only solution is $\boxed{x = 0}$.

7. $5\sqrt{x}+1 = 3\sqrt{x}+17$

Solution: This equation is a bit different. Although there are two radical expressions, they are identical. This equation is linear in $\sqrt{x}$. We can solve for $\sqrt{x}$ and then for x.

$$\begin{aligned}
5\sqrt{x}+1 &= 3\sqrt{x}+17 && \text{subtract } 3\sqrt{x}\\
2\sqrt{x}+1 &= 17 && \text{subtract } 1\\
2\sqrt{x} &= 16 && \text{divide by } 2\\
\sqrt{x} &= 8 && \text{square both sides}\\
x &= 64
\end{aligned}$$

We check: If $x = 64$, then

$$\text{LHS} = 5\sqrt{64}+1 = 5(8)+1 = 41 \text{ and } \text{RHS} = 3\sqrt{64}+17 = 3(8)+17 = 24+17 = 41 \checkmark$$

Thus $x = \boxed{64}$ is indeed a solution.

8. $\sqrt{2x+5}+5=x$

Solution:

$$\begin{aligned}
\sqrt{2x+5}+5 &= x && \text{subtract 5}\\
\sqrt{2x+5} &= x-5 && \text{square}\\
2x+5 &= x^2-10x+25 && \text{reduce one side to zero}\\
0 &= x^2-12x+20 && \text{factor}\\
0 &= (x-2)(x-10) && \Longrightarrow\ x_1=2 \text{ and } x_2=10
\end{aligned}$$

We check: If $x=2$, then

$$\text{LHS}=\sqrt{2(2)+5}+5=\sqrt{4+5}+5=\sqrt{9}+5=3+5=8 \text{ and } \text{RHS}=2 \qquad \text{RHS}\neq\text{LHS}$$

Thus $x=2$ is NOT a solution. If $x=10$, then

$$\text{LHS}=\sqrt{2(10)+5}+5=\sqrt{20+5}+5=\sqrt{25}+5=5+5=10=\text{RHS}\ \checkmark$$

Thus $\boxed{x=10}$ is the only solution.

9. $\sqrt{2x+5}-\sqrt{x-1}=\sqrt{x+2}$

Solution: This equation contains three different radical expressions. Our method still works.

$$\begin{aligned}
\sqrt{2x+5}-\sqrt{x-1} &= \sqrt{x+2} && \text{add } \sqrt{x-1}\\
\sqrt{2x+5} &= \sqrt{x+2}+\sqrt{x-1} && \text{square}\\
\left(\sqrt{2x+5}\right)^2 &= \left(\sqrt{x+2}+\sqrt{x-1}\right)^2 &&\\
2x+5 &= \left(\sqrt{x+2}+\sqrt{x-1}\right)\left(\sqrt{x+2}+\sqrt{x-1}\right) &&\\
2x+5 &= \underbrace{\sqrt{x+2}\sqrt{x+2}}_{\text{F}}+\underbrace{\sqrt{x+2}\sqrt{x-1}}_{\text{O}}+\underbrace{\sqrt{x-1}\sqrt{x+2}}_{\text{I}}+\underbrace{\sqrt{x-1}\sqrt{x-1}}_{\text{L}} &&\\
2x+5 &= x+2+2\sqrt{x-1}\sqrt{x+2}+x-1 &&\\
2x+5 &= 2x+1+2\sqrt{x-1}\sqrt{x+2} && \text{subtract } 2x\\
5 &= 1+2\sqrt{(x-1)(x+2)} && \text{subtract 1}\\
4 &= 2\sqrt{(x-1)(x+2)} && \text{divide by 2}\\
2 &= \sqrt{(x-1)(x+2)} && \text{square}\\
4 &= (x-1)(x+2) && \text{FOIL}\\
4 &= x^2+x-2 && \text{reduce one seide to zero}\\
0 &= x^2+x-6 && \text{factor}\\
0 &= (x+3)(x-2) && \Longrightarrow\ x_1=-3 \text{ and } x_2=2
\end{aligned}$$

We check: If $x=-3$, then

$$\text{LHS}=\sqrt{2(-3)+5}-\sqrt{(-3)-1}=\sqrt{-1}-\sqrt{-4}=\text{undefined}$$

Since the left hand side is undefined, $x=-3$ is NOT a solution. If $x=2$, then

$$\text{LHS}=\sqrt{2(2)+5}-\sqrt{2-1}=\sqrt{9}-\sqrt{1}=3-1=2=\text{RHS}\ \checkmark$$

Thus $\boxed{x=2}$ is the only solution.

10. $\sqrt{10x-1}+2=-x$

Solution:

$$\begin{array}{rcll}\sqrt{10x-1}+2 & = & -x & \text{subtract } 2\\ \sqrt{10x-1} & = & -2-x & \text{square}\\ 10x-1 & = & x^2+4x+4 & \text{subtract } 10x\text{, add } 1\\ 0 & = & x^2-6x+5 & \\ 0 & = & (x-1)(x-5) & \Longrightarrow \ x_1=1 \ \text{ and } \ x_2=1\end{array}$$

We check both answers: if $x=1$, then

$$\text{LHS}=\sqrt{10\cdot 1-1}=\sqrt{9}=3 \quad \text{and} \quad \text{RHS}=-1 \qquad \text{LHS}\neq\text{RHS}$$

thus 1 does not work. If $x=5$, then

$$\text{LHS}=\sqrt{10\cdot 5-1}=\sqrt{49}=7 \quad \text{and} \quad \text{RHS}=-5 \qquad \text{LHS}\neq\text{RHS}$$

thus 5 does not work either. This equation has no real solution.

11. $\sqrt{3x+1}-\sqrt{x-4}=3$

Solution:

$$\begin{array}{rcll}\sqrt{3x+1}-\sqrt{x-4} & = & 3 & \text{add } \sqrt{x-4} \text{ to both sides}\\ \sqrt{3x+1} & = & 3+\sqrt{x-4} & \text{square both sides}\\ 3x+1 & = & \left(3+\sqrt{x-4}\right)^2 & \\ 3x+1 & = & \left(3+\sqrt{x-4}\right)\left(3+\sqrt{x-4}\right) & \\ 3x+1 & = & 9+3\sqrt{x-4}+3\sqrt{x-4}+x-4 & \\ 3x+1 & = & x+5+6\sqrt{x-4} & \text{subtract } x\\ 2x+1 & = & 5+6\sqrt{x-4} & \text{subtract } 5\\ 2x-4 & = & 6\sqrt{x-4} & \\ 2(x-2) & = & 6\sqrt{x-4} & \text{divide by } 2\\ x-2 & = & 3\sqrt{x-4} & \text{square both sides}\\ (x-2)^2 & = & 9(x-4) & \text{FOIL, distribute}\\ x^2-4x+4 & = & 9x-36 & \text{reduce one side to zero}\\ x^2-13x+40 & = & 0 & \text{factor}\\ (x-5)(x-8) & = & 0 & \Longrightarrow \ x_1=5 \ \text{ and } \ x_2=8\end{array}$$

We check: if $x=5$, then

$$\text{LHS}=\sqrt{3(5)+1}-\sqrt{5-4}=\sqrt{16}-\sqrt{1}=4-1=3=\text{RHS}\ \checkmark$$

If $x=8$, then

$$\text{LHS}=\sqrt{3(8)+1}-\sqrt{8-4}=\sqrt{25}-\sqrt{4}=5-2=3=\text{RHS}\ \checkmark$$

The solutions are 5 and 8.

12. $\sqrt{x+10}+10=x$

Solution:

$$\begin{aligned}
\sqrt{x+10}+10 &= x && \text{subtract } 10\\
\sqrt{x+10} &= x-10 && \text{square}\\
x+10 &= (x-10)^2 \\
x+10 &= x^2-20x+100 && \text{reduce one side to zero}\\
0 &= x^2-21x+90 && \text{factor}\\
0 &= (x-6)(x-15) && \Longrightarrow \ x_1=6 \ \text{ and } \ x_2=15
\end{aligned}$$

We check: if $x=6$, then

$$\text{LHS}=\sqrt{6+10}+10=\sqrt{16}+10=4+10=14 \ \text{ and } \ \text{RHS}=6 \qquad \text{LHS} \neq \text{RHS}$$

and if $x=15$, then

$$\text{LHS}=\sqrt{15+10}+10=\sqrt{25}+10=5+10=15=\text{RHS} \ \checkmark$$

since $x=6$ doesn't work, the only solution is $\boxed{15}$.

13. $\sqrt{4x-11}=\sqrt{x-1}+\sqrt{x-4}$

Solution:

$$\begin{aligned}
\sqrt{4x-11} &= \sqrt{x-1}+\sqrt{x-4} && \text{square}\\
\left(\sqrt{4x-11}\right)^2 &= \left(\sqrt{x-1}+\sqrt{x-4}\right)^2 \\
4x-11 &= \left(\sqrt{x-1}+\sqrt{x-4}\right)\left(\sqrt{x-1}+\sqrt{x-4}\right) && \text{FOIL}\\
4x-11 &= \underbrace{\sqrt{x-1}\sqrt{x-1}}_{\text{F}}+\underbrace{\sqrt{x-1}\sqrt{x-4}}_{\text{O}}+\underbrace{\sqrt{x-4}\sqrt{x-1}}_{\text{I}}+\underbrace{\sqrt{x-4}\sqrt{x-4}}_{\text{L}} \\
4x-11 &= x-1+2\sqrt{x-1}\sqrt{x-4}+x-4 && \text{combine like terms}\\
4x-11 &= 2x-5+2\sqrt{x-1}\sqrt{x-4} && \text{subtract } 2x\\
2x-11 &= -5+2\sqrt{x-1}\sqrt{x-4} && \text{add } 5\\
2x-6 &= 2\sqrt{x-1}\sqrt{x-4} \\
2(x-3) &= 2\sqrt{x-1}\sqrt{x-4} && \text{divide by } 2\\
x-3 &= \sqrt{(x-1)(x-4)} && \text{square}\\
(x-3)^2 &= (x-1)(x-4) && \text{FOIL}\\
x^2-6x+9 &= x^2-5x+4 && \text{subtract } x^2\\
-6x+9 &= -5x+4 && \text{add } 6x\\
9 &= x+4 && \text{subtract } 4\\
5 &= x
\end{aligned}$$

We check:

$$\text{LHS}=\sqrt{5-1}+\sqrt{5-4}=\sqrt{4}+\sqrt{1}=2+1=3 \ \text{ and } \ \text{RHS}=\sqrt{4x-11}=\sqrt{4(5)-11}=\sqrt{9}=3$$

And so the solution is $\boxed{5}$.

14. $\sqrt{4x+6} = \sqrt{x+1} - \sqrt{x+5}$

Solution:

$$\begin{aligned}
\sqrt{4x+6} &= \sqrt{x+1} - \sqrt{x+5} && \text{square}\\
\left(\sqrt{4x+6}\right)^2 &= \left(\sqrt{x+1} - \sqrt{x+5}\right)^2 &&\\
4x+6 &= \underbrace{\sqrt{x+1}\sqrt{x+1}}_{\text{F}} - \underbrace{\sqrt{x+1}\sqrt{x+5}}_{\text{O}} - \underbrace{\sqrt{x+5}\sqrt{x+1}}_{\text{I}} + \underbrace{\sqrt{x+5}\sqrt{x+5}}_{\text{L}} &&\\
4x+6 &= x+1-2\sqrt{(x+1)(x+5)}+x+5 &&\\
4x+6 &= 2x+6-2\sqrt{(x+1)(x+5)} && \text{subtract } 2x\\
2x+6 &= 6-2\sqrt{(x+1)(x+5)} && \text{subtract } 6\\
2x &= -2\sqrt{(x+1)(x+5)} && \text{divide by } 2\\
x &= -\sqrt{(x+1)(x+5)} && \text{square}\\
x^2 &= \left(-\sqrt{(x+1)(x+5)}\right)^2 &&\\
x^2 &= (x+1)(x+5) && \text{FOIL right hand side}\\
x^2 &= x^2+6x+5 && \text{subtract } x^2\\
0 &= 6x+5 && \text{subtract } 5\\
-5 &= 6x && \text{divide by } 6\\
-\frac{5}{6} &= x &&
\end{aligned}$$

We check: if $x = -\dfrac{5}{6}$, then

$$\begin{aligned}
\text{LHS} &= \sqrt{4\left(-\frac{5}{6}\right)+6} = \sqrt{-\frac{10}{3}+6} = \sqrt{-\frac{10}{3}+\frac{18}{3}} = \sqrt{\frac{8}{3}}\\
\text{RHS} &= \sqrt{-\frac{5}{6}+1} - \sqrt{-\frac{5}{6}+5} = \sqrt{\frac{1}{6}} - \sqrt{\left(-\frac{5}{6}\right)+\frac{30}{6}} = \sqrt{\frac{1}{6}} - \sqrt{\frac{25}{6}} = \frac{\sqrt{1}}{\sqrt{6}} - \frac{\sqrt{25}}{\sqrt{6}}\\
&= \frac{1}{\sqrt{6}} - \frac{5}{\sqrt{6}} = \frac{1-5}{\sqrt{6}} = -\frac{4}{\sqrt{6}}
\end{aligned}$$

Since the left hand side is positive, and the right hand side is negative, these two numbers can not be equal. This equation has no real solution.

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