Sample Problems

1. Simplify each of the following.

   a) $\dfrac{2a-5}{5-2a}$ b) $\dfrac{x^3-x}{x+1}$ c) $\dfrac{2x+1}{4x^2-1}$ d) $\dfrac{x^2-30x-675}{x^2-6x-1755}$ e) $\dfrac{(x+5)-2}{5(x+2)-(x-2)}$

2. Perform the indicated operations and simplify.

   a) $\dfrac{c}{5a}\cdot\dfrac{15a^2b}{3b^2c}$ c) $\dfrac{x^2-3x}{x^2-8x+15}\cdot\dfrac{x^2-16x+15}{x^2-x}$ e) $\dfrac{x^2-10x+25}{x^2-10x+24}\left(\dfrac{x^2-2x-8}{x^2-6x+5}\div\dfrac{x-5}{x-1}\right)$

   b) $\dfrac{5x-30}{x^2-36}\cdot\dfrac{3x+18}{5}$ d) $\dfrac{x^2-9}{x^2-4x-21}\div\dfrac{4x-12}{3x-21}$

3. Perform the indicated operations and simplify.

   a) $\dfrac{3x}{x-2}-\dfrac{x+4}{x-2}$ c) $\dfrac{1}{x-y}-\dfrac{1}{x+y}$ e) $\dfrac{x^2-5x+78}{18x+x^2-208}-\dfrac{x}{x+26}$

   b) $\dfrac{10}{x-y}-\dfrac{5}{y-x}$ d) $\dfrac{2}{p-5}-\dfrac{p+11}{p^2-2p-15}$ f) $\left[\left(\dfrac{1}{b}-b\right)\div\left(1-\dfrac{1}{b}\right)\right](1+b)$

4. Solve the following equation. Make sure to check your solution(s).

$$\frac{-3x}{x+1}+\frac{4x+1}{x}=\frac{-3}{x^2+x}$$

5. Suppose that a, b and x are non-zero real numbers, and $\dfrac{1}{x}=\dfrac{1}{a}-\dfrac{1}{b}$. Solve this formula for x.

Practice Problems

1. Simplify each of the following.

   a) $\dfrac{2b-5}{10-4b}$ c) $\dfrac{4t^2-9}{4t-6}$ e) $\dfrac{6m+m^2-216}{6m-m^2+72}$ g) $\dfrac{-7x-11-3(x-2)}{5x-11+3(x+5)}$

   b) $\dfrac{x^2-1}{x+1}$ d) $\dfrac{p^2-p}{p^2-1}$ f) $\dfrac{x^2-144x-900}{x^2-158x+1200}$

2. Perform the indicated operations and simplify.

a) $\dfrac{x}{5yz} \cdot \dfrac{10x^2y^3z}{4xy^2}$

b) $\dfrac{a^2 - 8a + 16}{a} \cdot \dfrac{a^3}{4-a}$

c) $\dfrac{x^2 - 5x}{x^2 - 2x - 15} \cdot \dfrac{x^2 - 9}{x^2 - 3x}$

d) $\dfrac{x^2 - 4x - 21}{x^2 - 49} \div \dfrac{8x + x^2 + 15}{2x + x^2 - 35}$

e) $\dfrac{5y - 35}{y^2 - 2y - 35} \cdot \dfrac{3y + 15}{5y - 5}$

f) $\dfrac{2x^2 - 98}{x^2 - 6x - 7} \div \dfrac{21 + 3x}{6x^2 - 6}$

3. Perform the indicated operations and simplify.

a) $\dfrac{x+y}{2} - \dfrac{x-y}{2}$

b) $\dfrac{3a}{a-1} - \dfrac{a+2}{a-1}$

c) $\dfrac{a+1}{a-b} - \dfrac{b-1}{b-a}$

d) $\dfrac{5y+1}{y+5} + \dfrac{y^2-1}{y+5}$

e) $\dfrac{1}{p-5} - \dfrac{1}{p+5}$

f) $\dfrac{x-5}{x+2} - \dfrac{3}{2-x} - \dfrac{14-x}{x^2-4}$

g) $\dfrac{3}{x+1} - \dfrac{2}{x-1} + \dfrac{x+3}{x^2-1}$

h) $\dfrac{2(m+2)}{4m + m^2 - 12} - \dfrac{1}{m-2}$

i) $\dfrac{2a - \dfrac{1}{8a}}{4 + \dfrac{1}{a}}$

4. Solve each of the following equations. Make sure to check your solutions.

a) $\dfrac{4-x}{x} + \dfrac{3}{2} = -\dfrac{4}{x}$

b) $\dfrac{2}{x-1} + \dfrac{3}{4} = \dfrac{5}{x-1}$

c) $\dfrac{2}{a+1} + \dfrac{5}{a-1} = \dfrac{3}{a^2-1}$

d) $\dfrac{2}{a+1} + \dfrac{5}{a-1} = \dfrac{10}{a^2-1}$

e) $\dfrac{x}{x+3} + \dfrac{5}{x-7} = \dfrac{30}{x^2 - 4x - 21}$

f) $\dfrac{x}{x-2} - \dfrac{4}{x+2} = \dfrac{8}{x^2-4}$

5. a) Suppose that a, b and x are non-zero real numbers, and $\dfrac{5}{x} = \dfrac{2}{a} + \dfrac{3}{b}$. Solve this formula for x.

b) Solve $\dfrac{1}{k} + \dfrac{3}{r} = \dfrac{5}{z}$ for r.

Sample Problems – Answers

1\. a) -1 b) $x^2 - x$ c) $\dfrac{1}{2x-1}$ d) $\dfrac{x+15}{x+39}$ e) $\dfrac{1}{4}$

2\. a) $\dfrac{a}{b}$ b) 3 c) $\dfrac{x-15}{x-5}$ d) $\dfrac{3}{4}$ e) $\dfrac{x+2}{x-6}$

3\. a) 2 b) $\dfrac{15}{x-y}$ or $-\dfrac{15}{y-x}$ c) $\dfrac{2y}{(x-y)(x+y)}$ or $\dfrac{2y}{x^2-y^2}$ d) $\dfrac{1}{p+3}$

e) $\dfrac{3}{x-8}$ f) $-(b+1)^2$

4\. -4

5\. $x = \dfrac{ab}{b-a}$

Practice Problems – Answers

1\. a) $-\dfrac{1}{2}$ b) $x-1$ c) $\dfrac{2t+3}{2}$ d) $\dfrac{p}{p+1}$ e) $-\dfrac{m+18}{m+6}$ f) $\dfrac{x+6}{x-8}$ g) $-\dfrac{5}{4}$

2\. a) $\dfrac{x^2}{2}$ b) $-a^2(a-4)$ c) 1 d) $\dfrac{x-5}{x+5}$ e) $\dfrac{3}{y-1}$ f) $4x-4$

3\. a) y b) 2 c) $\dfrac{a+b}{a-b}$ d) y e) $\dfrac{10}{p^2-25}$ f) $\dfrac{x-1}{x+2}$ g) $\dfrac{2}{x+1}$ h) $\dfrac{1}{m+6}$ i) $\dfrac{4a-1}{8}$

4\. a) -16 b) 5 c) 0 d) no solution e) 5 f) 0

5\. a) $x = \dfrac{5ab}{3a+2b}$ b) $r = \dfrac{-3kz}{z-5k}$

Sample Problems – Solutions

1. Simplify each of the following.

a) $\dfrac{2a-5}{5-2a} = \quad -1$

Solution: We need to notice that the numerator and denominator are opposites of each other. Indeed, the opposite of $2a-5$ is $5-2a$ since

$$-1\,(2a-5) = -2a+5 = 5-2a$$

Thus

$$\frac{2a-5}{5-2a} = \frac{2a-5}{-1\,(2a-5)} = \frac{1}{-1} = -1$$

b) $\dfrac{x^3-x}{x+1} = \quad x^2-x$

Solution: In general, we factor both numerator and denominator and then simplify. In this case we only factor the numerator, since the denominator is too small to factor. After we factor out the greatest common factor (or GCF) which is x, the expression factors via the difference of squares theorem.

$$x^3-x = x\left(x^2-1\right) = x\,(x+1)\,(x-1)$$

Then we simplify the fraction by canceling out the same factor from numerator and denominator.

$$\frac{x^3-x}{x+1} = \frac{x\,(x+1)\,(x-1)}{x+1} = x\,(x-1) \quad \text{or} \quad x^2-x$$

c) $\dfrac{2x+1}{4x^2-1} = \quad \dfrac{1}{2x-1}$

Solution: We factor the denominator via the difference of squares theorem, and then cancel.

$$\frac{2x+1}{4x^2-1} = \frac{2x+1}{(2x+1)\,(2x-1)} = \frac{1}{2x-1}$$

d) $\dfrac{x^2-30x-675}{x^2-6x-1755} = \quad \dfrac{x+15}{x+39}$

Solution: We factor both numerator and denominator and then simplify. We can easily factor both of these polynomials by completing the square.

$$\frac{x^2-30x-675}{x^2-6x-1755} = \frac{(x+15)\,(x-45)}{(x+39)\,(x-45)} = \frac{x+15}{x+39}$$

e) $\dfrac{(x+5)-2}{5\,(x+2)-(x-2)} = \quad \dfrac{1}{4}$

Solution: We simplify both numerator and denominator, then if possible, factor these and then simplify the fraction by cancellation.

$$\frac{(x+5)-2}{5\,(x+2)-(x-2)} = \frac{x+5-2}{5x+10-x+2} = \frac{x+3}{4x+12} = \frac{x+3}{4\,(x+3)} = \frac{1}{4}$$

2. Perform the indicated operations and simplify.

a) $\dfrac{c}{5a} \cdot \dfrac{15a^2b}{3b^2c} = \quad \dfrac{a}{b}$

Solution: We perform the multiplication among fractions (top by top, bottom by bottom) and then simplify by canceling factors appearing in both the numerator and denominator.

$$\frac{c}{5a} \cdot \frac{15a^2b}{3b^2c} = \frac{15a^2bc}{15ab^2c} = \frac{a}{b}$$

b) $\dfrac{5x-30}{x^2-36} \cdot \dfrac{3x+18}{5} = \quad 3$

Solution: we will factor whatever we can and then simplify by canceling factors appearing in both the numerator and denominator.

$$\frac{5x-30}{x^2-36} \cdot \frac{3x+18}{5} = \frac{5(x-6)}{(x+6)(x-6)} \cdot \frac{3(x+6)}{5} = 3$$

c) $\dfrac{x^2-3x}{x^2-8x+15} \cdot \dfrac{x^2-16x+15}{x^2-x} = \quad \dfrac{x-15}{x-5}$

Solution: We factor whatever we can and then simplify by canceling factors appearing in both the numerator and denominator. We can factor all of these polynomials by completing the square or by factoring out the greatest common factor.

$$\frac{x^2-3x}{x^2-8x+15} \cdot \frac{x^2-16x+15}{x^2-x} = \frac{x(x-3)}{(x-3)(x-5)} \cdot \frac{(x-1)(x-15)}{x(x-1)} = \frac{x}{x-5} \cdot \frac{x-15}{x} = \frac{x-15}{x-5}$$

d) $\dfrac{x^2-9}{x^2-4x-21} \div \dfrac{4x-12}{3x-21} = \quad \dfrac{3}{4}$

Solution: We first re-write the division as multiplication by the reciprocal.

$$\frac{x^2-9}{x^2-4x-21} \div \frac{4x-12}{3x-21} = \frac{x^2-9}{x^2-4x-21} \cdot \frac{3x-21}{4x-12}$$

We now factor the polynomials appearing in the fractions

$$\begin{aligned} x^2-9 &= (x+3)(x-3) & \qquad 4x-12 = 4(x-3) \\ x^2-4x-21 &= (x+3)(x-7) & \qquad 3x-21 = 3(x-7) \end{aligned}$$

We now re-write the fractions using these factored forms, and cancel out factors appearing in both numerator and denominator of the product.

$$\frac{x^2-9}{x^2-4x-21} \cdot \frac{3x-21}{4x-12} = \frac{(x+3)(x-3)}{(x+3)(x-7)} \cdot \frac{3(x-7)}{4(x-3)} = \frac{3(x+3)(x-3)(x-7)}{4(x+3)(x-3)(x-7)} = \frac{3}{4}$$

e) $\dfrac{x^2-10x+25}{x^2-10x+24}\left(\dfrac{x^2-2x-8}{x^2-6x+5} \div \dfrac{x-5}{x-1}\right) = \quad \dfrac{x+2}{x-6}$

Solution: We first re-write the division as multiplication by the reciprocal.

$$\frac{x^2-10x+25}{x^2-10x+24}\left(\frac{x^2-2x-8}{x^2-6x+5} \div \frac{x-5}{x-1}\right) = \frac{x^2-10x+25}{x^2-10x+24}\left(\frac{x^2-2x-8}{x^2-6x+5} \cdot \frac{x-1}{x-5}\right)$$

We now factor the polynomials appearing in each fraction.

$$\begin{aligned} x^2-10x+25 &= (x-5)^2 & \qquad x^2-2x-8 = (x+2)(x-4) \\ x^2-10x+24 &= (x-4)(x-6) & \qquad x^2-6x+5 = (x-1)(x-5) \end{aligned}$$

We now re-write the problem, using the factored form of polynomials.

$$\frac{x^2-10x+25}{x^2-10x+24}\left(\frac{x^2-2x-8}{x^2-6x+5}\cdot\frac{x-1}{x-5}\right)=\frac{(x-5)^2}{(x-4)(x-6)}\left(\frac{(x+2)(x-4)}{(x-1)(x-5)}\cdot\frac{x-1}{x-5}\right)$$

We now perform the multiplication within the parentheses. Notice that we can cancel out $x-1$.

$$\frac{(x-5)^2}{(x-4)(x-6)}\left(\frac{(x+2)(x-4)}{(x-1)(x-5)}\cdot\frac{x-1}{x-5}\right)=\frac{(x-5)^2}{(x-4)(x-6)}\cdot\frac{(x+2)(x-4)}{(x-5)^2}$$

We can now cancel out $(x-5)^2$ and $x-4$, and the final answer is $\dfrac{x+2}{x-6}$.

3. Perform the indicated operations and simplify.

a) $\dfrac{3x}{x-2}-\dfrac{x+4}{x-2}=\ 2$

Solution: This is a subtraction of fractions. The denominators are the same, the only difficulty is that we are subtracting expressions instead of numbers. The second pair of parentheses is essential.

$$\frac{3x}{x-2}-\frac{x+4}{x-2}=\frac{(3x)-(x+4)}{x-2}=\frac{3x-x-4}{x-2}=\frac{2x-4}{x-2}=\frac{2(x-2)}{x-2}=2$$

b) $\dfrac{10}{x-y}-\dfrac{5}{y-x}=\ \dfrac{15}{x-y}$ or $-\dfrac{15}{y-x}$

Solution: This problem is much easier when we realize that the denominators are opposites of each other, since the opposite of $y-x$ is $-1(y-x)=-y+x=x-y$

$$\frac{10}{x-y}-\frac{5}{y-x}=\frac{10}{x-y}-\frac{(-1)5}{(-1)(y-x)}=\frac{10}{x-y}-\frac{-5}{x-y}=\frac{10-(-5)}{x-y}=\frac{15}{x-y}$$

c) $\dfrac{1}{x-y}-\dfrac{1}{x+y}=\ \dfrac{2y}{(x-y)(x+y)}$ or $\dfrac{2y}{x^2-y^2}$

Solution: We first bring the two fractions to the common denominator, which is $(x+y)(x-y)=x^2-y^2$

$$\begin{aligned}\frac{1}{x-y}-\frac{1}{x+y}&=\frac{1\cdot(x+y)}{(x-y)(x+y)}-\frac{1\cdot(x-y)}{(x+y)(x-y)}\\&=\frac{x+y}{(x-y)(x+y)}-\frac{x-y}{(x-y)(x+y)}\end{aligned}$$

We are now ready to subtract. We need to be extremely careful to subtract the ENTIRE expression, and not just its first term. To do that, we need to insert parentheses around these expressions.

$$\frac{(x+y)-(x-y)}{(x-y)(x+y)}=\frac{x+y-x+y}{(x-y)(x+y)}=\frac{2y}{(x-y)(x+y)}\quad\text{or}\quad\frac{2y}{x^2-y^2}$$

The final answer can be presented as $\dfrac{2y}{(x-y)(x+y)}$ or $\dfrac{2y}{x^2-y^2}$. They are both equally correct.

d) $\dfrac{2}{p-5}-\dfrac{p+11}{p^2-2p-15}=\ \dfrac{1}{p+3}$

Solution: The denominator of the second fraction factors. $p^2-2p-15=(p+3)(p-5)$. We will now bring these fractions to the common denominator.

$$\frac{2}{p-5}-\frac{p+11}{p^2-2p-15}=\frac{2(p+3)}{(p-5)(p+3)}-\frac{p+11}{(p-5)(p+3)}$$

We can now perform the subtraction of polynomials in the numerator

$$\frac{2(p+3)-(p+11)}{(p-5)(p+3)}=\frac{2p+6-p-11}{(p-5)(p+3)}=\frac{p-5}{(p-5)(p+3)}=\frac{1}{p+3}$$

e) $\dfrac{x^2-5x+78}{18x+x^2-208}-\dfrac{x}{x+26}=\quad \dfrac{3}{x-8}$

Solution: The denominator of the first fraction factors. $x^2+18x-208=(x+26)(x-8)$. We will now bring these fractions to the common denominator.

$$\frac{x^2-5x+78}{x^2+18x-208}-\frac{x}{x+26}=\frac{x^2-5x+78}{(x+26)(x-8)}-\frac{x(x-8)}{(x+26)(x-8)}$$

We can now perform the subtraction of polynomials in the numerator

$$\frac{\left(x^2-5x+78\right)-x(x-8)}{(x+26)(x-8)}=\frac{x^2-5x+78-x^2+8x}{(x+26)(x-8)}=\frac{3x+78}{(x+26)(x-8)}$$

We are not done yet: after we factor out the greatest common factor from the numerator, the result simplifies.

$$\frac{3x+78}{(x+26)(x-8)}=\frac{3(x+26)}{(x+26)(x-8)}=\frac{3}{x-8}$$

f) $\left[\left(\dfrac{1}{b}-b\right)\div\left(1-\dfrac{1}{b}\right)\right](1+b)=\quad -(b+1)^2$

Solution: Let us perform the subtractions first.

$$\frac{1}{b}-b=\frac{1}{b}-\frac{b}{1}=\frac{1}{b}-\frac{b(b)}{1(b)}=\frac{1}{b}-\frac{b^2}{b}=\frac{1-b^2}{b}\quad\text{and}$$

$$1-\frac{1}{b}=\frac{1}{1}-\frac{1}{b}=\frac{1(b)}{1(b)}-\frac{1}{b}=\frac{b}{b}-\frac{1}{b}=\frac{b-1}{b}$$

Thus

$$\left[\left(\frac{1}{b}-b\right)\div\left(1-\frac{1}{b}\right)\right](1+b)=\left[\frac{1-b^2}{b}\div\frac{b-1}{b}\right](1+b)$$

We re-write the division as multiplication by the reciprocal, and then cancel.

$$\left[\frac{1-b^2}{\cancel{b}}\cdot\frac{\cancel{b}}{b-1}\right](1+b)=\frac{1-b^2}{b-1}(1+b)$$

This expression can be further simplified, since $1-b^2=(1+b)(1-b)$ via the difference of squares theorem, and $1-b$ and $b-1$ are opposites.

$$\frac{1-b^2}{b-1}(1+b)=\frac{(1-b)(1+b)}{b-1}(1+b)=\frac{-1(b-1)(1+b)}{b-1}(1+b)=-(b+1)^2$$

4. Solve the following equation. Make sure to check your solution(s). -4

$$\frac{-3x}{x+1}+\frac{4x+1}{x}=\frac{-3}{x^2+x}$$

Solution: We will bring both sides to the common denominator, which is $x(x+1)=x^2+x$. Notice that based on the first sight of this equation, 0 and -1 can not be solutions of this equation because they can not be substituted into the first line in a meaningful way.

$$\begin{aligned}
\frac{-3x}{x+1}+\frac{4x+1}{x} &= \frac{-3}{x^2+x} \\
\frac{-3x\,(x)}{(x+1)\,(x)}+\frac{(4x+1)\,(x+1)}{(x)\,(x+1)} &= \frac{-3}{x\,(x+1)} \qquad \text{we multiply by } x\,(x+1) \\
-3x^2+(4x+1)\,(x+1) &= -3 \\
-3x^2+\left(4x^2+4x+x+1\right) &= -3 \\
-3x^2+4x^2+5x+1 &= -3 \\
x^2+5x+4 &= 0 \\
(x+4)\,(x+1) &= 0 \quad \Longrightarrow \quad x_1=-4 \qquad x_2=-1
\end{aligned}$$

We check: if $x=-4$, then

$$\begin{aligned}
\text{LHS} &= \frac{-3\,(-4)}{-4+1}+\frac{4\,(-4)+1}{-4}=\frac{12}{-3}+\frac{-15}{-4}=-4+\frac{15}{4}=\frac{-16}{4}+\frac{15}{4}=-\frac{1}{4} \\
\text{RHS} &= \frac{-3}{(-4)^2+(-4)}=\frac{-3}{16-4}=\frac{-3}{12}=-\frac{1}{4}
\end{aligned}$$

and if $x=-1$, then

$$\text{LHS}=\frac{-3\,(-1)}{-1+1}+\frac{4\,(-1)+1}{-1}=\frac{3}{0}+\frac{-3}{-1}=\text{undefined}$$

since we would be dividing by zero. Consequently, only -4 is a solution of this equation.

5. Suppose that a, b and x are non-zero real numbers, and $\dfrac{1}{x}=\dfrac{1}{a}-\dfrac{1}{b}$. Solve this formula for x. $\dfrac{ab}{b-a}$

Solution: Method 1. Bringing to the common denominator first and then multiply both sides by that common denominator.

$$\begin{aligned}
\frac{1}{x} &= \frac{1}{a}-\frac{1}{b} \qquad \text{common denominator is } abx \\
\frac{1\,(ab)}{x\,(ab)} &= \frac{1\,(bx)}{a\,(bx)}-\frac{1\,(ax)}{b\,(ax)} \\
\frac{ab}{abx} &= \frac{bx-ax}{abx} \qquad \text{multiply both sides by } abx \\
ab &= bx-ax \qquad \text{factor out } x \\
ab &= x\,(b-a) \qquad \text{divide by } b-a \\
\frac{ab}{b-a} &= x
\end{aligned}$$

Method 2. Multiplying both sides by the common denominator.

$$\begin{aligned}
\frac{1}{x} &= \frac{1}{a}-\frac{1}{b} \qquad \text{multiply both sides by } abx \\
(abx)\,\frac{1}{x} &= (abx)\left(\frac{1}{a}-\frac{1}{b}\right) \qquad \text{distribute } abx \\
(abx)\,\frac{1}{x} &= (abx)\,\frac{1}{a}-(abx)\,\frac{1}{b} \qquad \text{cancel} \\
ab &= bx-ax \qquad \text{factor out } x \\
ab &= x\,(b-a) \qquad \text{divide by } b-a \\
\frac{ab}{b-a} &= x
\end{aligned}$$

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