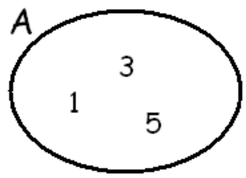
Definition: A **set** is a collection of objects. Two sets are equal if they contain the same objects.

Sets are usually denoted by uppercase letters. There are several ways a set could be given. We can describe a set using English language. In case of small sets, we can also simply list its elements, separated by a comma and enclosed in braces $\{$ and $\}$ can be used to denote set.

Example 1. Let M be the set of all one-digit natural numbers. Re-write this set by listing its elements.

We use the braces and list all one-digit natural numbers. $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

We can also describe a set using a diagram.
We depicted $A = \{1, 3, 5\}$.



Some famous sets have their own set theory label. For example, we already know the infinite set $\{1, 2, 3, ...\}$. This is the set of all natural numbers or counting numbers, and it is denoted by $\mathbb{N}$.

Sometimes we need to be more descriptive when specifying sets.

Example 2. $S = \left\{x^2 : x \text{ is a natural number and } x \leq 5\right\}$.

We read this as S is a set containing x^2, where x is a natural number and x is less than or equal to 5. Of course, such a small set can be expressed much simpler, by listing its elements, as $S = \{1, 4, 9, 16, 25\}$. But this notation is often very useful when describing infinite sets.

Definition: A set is a collection of objects. The objects that make up the set are called the **elements** or **members** of the set, or that it **belongs** to the set.

Notation: If x is an element of a set S, we write by $x \in S$. If y is not an element of S, we write $y \notin S$.

Suppose that $A = \{1, 3, 5\}$. The following statements are all true.

$3 \in A$	read as: 3 belongs to A	$4 \notin A$	read as: 4 does not belong to A
$5 \in A$	read as: 5 is an element of A	$-6 \notin A$	read as: -6 is not an element of A

Example 3. Suppose that $A = \{1, 3, 5\}$ and that $\mathbb{N}$ is the set of all natural numbers, in short, $\mathbb{N} = \{1, 2, 3, ...\}$. Determine whether the given statements are true or false.

a) $1 \in A$ b) $2 \in A$ c) $-1 \in \mathbb{N}$ d) $5 \notin \mathbb{N}$ e) $4 \notin A$

Solution: a) The statement $1 \in A$ reads: 1 *belongs to set* A. This is true as 1 is an element of set A.

b) The statement $2 \in A$ reads: 2 *belongs to set* A. This is not true as 2 is not an element of set A.

c) The statement $-1 \in \mathbb{N}$ reads: -1 *belongs to the set of all natural numbers*.
(In short, -1 is a natural number). This statement is false.

d) The statement $5 \notin \mathbb{N}$ reads: 5 *does not belong to the set of all natural numbers*.
(In short, 5 is not a natural number). This statement is false.

e) The statement $4 \notin A$ reads: 4 *does not belong to set* A. This statement is true.

Definition: Two sets are **equal** if they contain the same objects. We use the symbol $=$ to denote equal sets.

When writing a set, the order of listing and repetition of its elements does not change a set.

Example 4. Let A be the set of odd natural numbers between 0 and 6. Suppose further that $B = \{5,\ 1,\ 3\}$, and that $C = \{5,\ 1,\ 5,\ 1,\ 1,\ 1,\ 3,\ 1\}$. Then all three sets are equal to each other, i.e $A = B = C$.

Since we are free to list the elements of a set any way we want to, it is often startegic to keep things organized by listing elements in increasing order. In this case, $A = B = C = \{1,\ 3,\ 5\}$. Sometimes we have reasons to part from this convention.

Definition: The **set of all integers**, denoted by $\mathbb{Z}$, is the set containing all natural numbers, their opposites, and zero.

$$\mathbb{Z} = \{0,\ 1,\ -1,\ 2,\ -2,\ 3,\ -3,\ 4,\ -4, \ldots\}$$

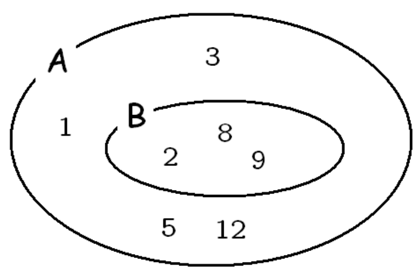
Some people prefer to present the set of all integers sort of organized, as $\mathbb{Z} = \{\ldots -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\}$. The disadvantage here is that both the beginning and the end of this infinite list goes on forever. More precisely, there is no beginning and no end. Both presentations are commonly used.

Definition: There is a unique set that contains no elements. It is called the **empty set** and is denoted by $\varnothing$ or by $\{\quad\}$.

Definition: Set B is a **subset** of set A if all elements of B also belong to A. Notation: $B \subseteq A$

There is another way to express this relationship. B is a subset of A if for all things x in the world, if x is an element of B, then x is also an element of A. This approach might be helpful later.

Example 5. Suppose that $A = \{1, 2, 3, 5, 8, 9, 12\}$ and $B = \{2, 8, 9\}$. Then B is a subset of A.



Example 6. Suppose that $X = \{a, b, d, f\}$ and $Y = \{a, b, c, d, e, f, g\}$. Then $X \subseteq Y$.

Example 7. Suppose that $S = \{1, 4, 9, 16\}$. Then $S \subseteq S$.

While this might look strange at first, the definition of subset applies. Every element of S is an element of S. Perhaps this statement is similar to $5 \leq 5$. For every number x, the statement $x \leq x$ is true.

Even more interestingly, the empty set is also a subset of every set. This is because the definition applies, even if strangely so. For every object x in the world, if x is in the empty set (it's not), then it in set A. We say that this statement is **vacuously true**.

Theorem: For all sets S, the following are both true: $\varnothing \subseteq S$ and $S \subseteq S$.

Example 8. If $\mathbb{N}$ is the set of all natural numbers and $\mathbb{Z}$ is the set of all integers, then $\mathbb{N} \subseteq \mathbb{Z}$.

Example 9. Suppose that E is the set of all even natural numbers, $E = \{2,\ 4,\ 6,\ 8,\ 10,\ \ldots\}$, and recall that $\mathbb{N} = \{1,\ 2,\ 3,\ \ldots\}$. Then $E \subseteq \mathbb{N}$.

Example 10. Suppose that L is the set of all letters in the English alphabet, and V is the set of vowels in the English alphabet. Then $V \subseteq L$.

Example 11. Let M be the set of all mammals and D the set of all dogs. Then D is a subset of M, or, in short, $D \subseteq M$.

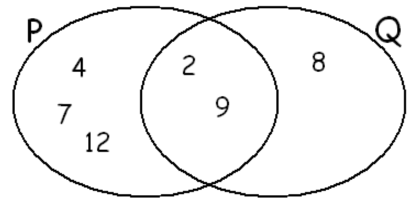
Definition: If A and B are sets, then the **intersection** of A and B, denoted by $A \cap B$, is the set such that for all x, $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

The intersection of two sets is the set of all elements that belong to both sets.

Example 12. Suppose that $P = \{2, 4, 7, 9, 12\}$ and $Q = \{2, 8, 9\}$. Find $P \cap Q$.

Solution: The intersection of P and Q is the set containing those elements that are in **both** P and Q. Since P and Q are small sets, we check from element to element, and collect those that belong to both. We can see that $P \cap Q = \{2, 9\}$.

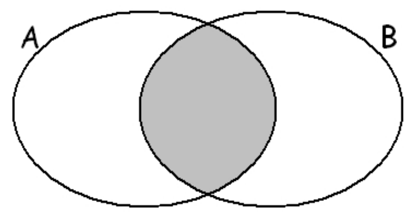
A picture such as this is called a Venn diagram. Venn diagrams often provide useful visual tools to solve set theory problems. We can depict the intersection using Venn Diagrams.



Example 13. Suppose that $A = \{1, 3, 5, 7, 9\}$ and $B = \{5, 6, 7, 8, 9, 10\}$. Find $A \cap B$.

Solution: The intersection of A and B is the set containing those elements that are in both A and B. Since A and B are small sets, we check from element to element, and collect those that belong to both. We can see that $A \cap B = \{5, 7, 9\}$.

If we use a Venn diagram, the intersection of the two sets is the 'overlap' between the two sets as shown.



The shaded region is $A \cap B$

Example 14. Suppose that $T = \{3, 4, 7, 10\}$ and $Q = \{1, 6, 8\}$. Find $T \cap Q$.

Solution: As we look for elements in common, we find none. Thus $T \cap Q = \varnothing$. When this happens, we say that the two sets are **disjoint**.

We want the intersection of two sets to always be a set. In other words, we want the set of sets to be closed under intersection. This is why it was important for us to define $\varnothing$, the empty set.

Example 15. Find $\mathbb{N} \cap \mathbb{Z}$.

Solution: If we start with the natural numbers, we notice that they are automatically in $\mathbb{Z}$. Indeed, $\mathbb{N}$ is a subset of $\mathbb{Z}$, and so every element in $\mathbb{N}$ is also in $\mathbb{Z}$ and thus in both sets. However, with the negative integers and zero we find that they are not in both sets because they are not in $\mathbb{N}$. Thus the intersection of the two sets is $\mathbb{N}$. In short, $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}$.

Definition: If A and B are sets, then the **union** of A and B, denoted by $A \cup B$, is the set such that for all x,
$x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

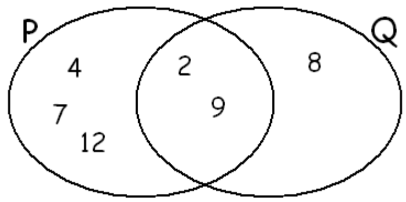
The word 'or' is used in the strict mathematical sense. $x \in A$ or $x \in B$ is true if either x is in A only, or x is in B only, or if x is in both. So, x is in the union of A and B if it is in A, B, or both.

The union of two sets is the set of all elements from one set, put together with the set of all elements of the other. Imagine we throw the elements of both sets together and then we list them as a set, ignoring repetitions.

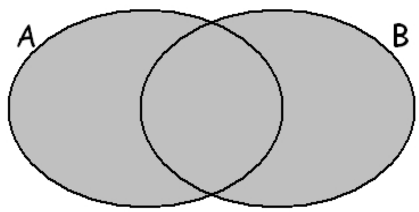
Example 16. Suppose that $P = \{2, 4, 7, 9, 12\}$ and $Q = \{2, 8, 9\}$. Find $P \cup Q$.

Solution: The union of P and Q is the set containing those elements that are in P or in Q. Since P and Q are small sets, we check from element to element, and collect those that belong to either sets or to both. Another way of visualizing the union is to throw together P and Q and list the resulting set without repetition. We can see that $P \cup Q = \{2, 4, 7, 8, 9, 12\}$.

A Venn diagram might help again. For the union, we collect every element from the three separate regions.



If we use a Venn diagram, the union of the two sets is the collection of those three regions as shown.



The shaded region is $A \cup B$

Example 17. Find $\mathbb{N} \cup \mathbb{Z}$.

Solution: Let us start with the integers this time. All integers are in the union since they are in $\mathbb{Z}$. Now we look at the other set, $\mathbb{N}$, and notice that all natural numbers are already listed in the union because they are automatically in $\mathbb{Z}$. Indeed, $\mathbb{N}$ is a subset of $\mathbb{Z}$, and so every element in $\mathbb{N}$ is also in $\mathbb{Z}$. Therefore, $\mathbb{N}$ does not bring anything new to the union, and so $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$.

Practice Problems

1. Suppose that S is a set defined as $S = \{-2, 4, 5, 16\}$ and recall that $\mathbb{N} = \{1, 2, 3, ...\}$. Determine whether the given statements are true or false.

   a) $-2 \in S$ b) $-2 \in \mathbb{N}$ c) $-3 \notin \mathbb{N}$ d) $5 \notin S$ e) $1 \in \mathbb{N}$

2. Let $A = \{1, 2, 5, 8, 9\}$ and $B = \{2, 4, 6, 8\}$.

   a) Draw a Venn diagram depicting these sets.

   b) Find each of the following.

   i) $A \cap B$ ii) $A \cup B$ iii) $B \cup (A \cap B)$

   c) Label each of the following statements as true or false.

   i) $A \subseteq A \cap B$ ii) $B \subseteq A \cup B$ iii) $A \cap B \subseteq A \cup B$

3. Let P denote the set of all students taking physics at Truman College. Let M denote the set of all students taking mathematics at Truman College.

   a) describe the set $P \cap M$ b) describe the set $P \cup M$

4. Label each of the following statements as true or false.

   a) $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}$ b) $\mathbb{N} \cap \mathbb{Z} = \mathbb{Z}$ c) $\mathbb{N} \cup \mathbb{Z} = \mathbb{N}$ d) $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$

5. Label each of the following statements as true or false. (Hint: make up suitable examples for yourself and then investigate!)

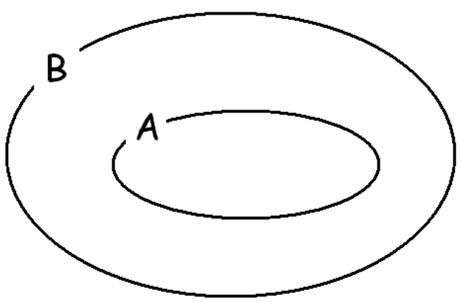
   a) If $A \subseteq B$, then $A \cap B = A$

   b) If $A \subseteq B$, then $A \cup B = B$

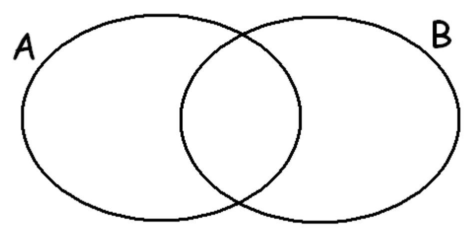
   c) For all sets A and B, $A \cap B \subseteq A$

   d) For all sets A and B, $B \subseteq A \cup B$

6. Recall that a visual representation of subset is to draw one set inside the other. However, this is not a Venn diagram. In case of a Venn diagram, we must have the three distinct regions.



This is not a Venn diagram.



This is a Venn diagram.

Given a Venn diagram depicting sets A and B, how does it show up that A is a subset of B?

Enrichment

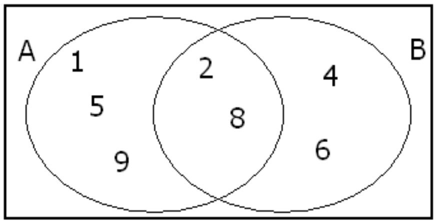
1. Suppose that A and B are sets such that $A \cap B = \{1, 2, 5\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$. How many different sets are possible for A?

2. Our junior class had 60 students. If 42 students took history, 35 students took French, and 19 took both history and French, how many students in the junior class took neither French nor history? (Hint: this is the kind of problem in which a Venn diagram can be very helpful.)

Answers

1. a) true b) false c) true d) false e) true

2. a)



b) i) $\{2, 8\}$ ii) $\{1, 2, 4, 5, 6, 8, 9\}$ iii) $\{2, 4, 6, 8\}$

c) i) false ii) true iii) true

3. a) $P \cap M$ - the set of all students taking mathematics and physics at Truman College.
b) $P \cup M$ - the set of all students taking mathematics or physics or both at Truman College.

4. a) true b) false c) false d) true

5. a) true b) true c) true d) true

6. If A is a subset of B, then there is no element that belongs to A but not to B. This will show up on the Venn diagram by the shaded region shown containing no elements.