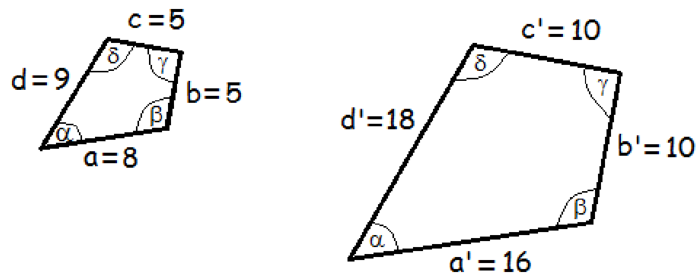
Similarity is a very important relation used in geometry.

> **Definition:** Two objects are **similar** to each other if they are either identical (also called congruent) or one is an enlargement of the other.

Similar objects always have the following properties: Their corresponding angles are the same, and their corresponding angles are proportional.

The two quadrilaterals shown are similar. When labeling the sides, we often suggest which side corresponds to which side. The side corresponding to a is often denoted by a'.



The corresponding sides are proportional. This can be approached in two different ways as follows.

1. We can express how one quadrilateral was enlarged to obtain the other. In this case, every side was doubled. We can express this magnifying factor as the same for all pairs of corresponding sides between the two quadrilaterals.

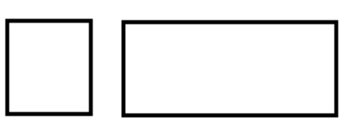
$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{d'}{d}$$

2. We can also express that from one quadrilateral to the other, the ratio between the sides was preserved.

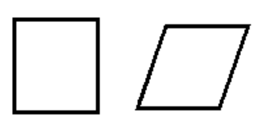
$$\frac{a}{b} = \frac{a'}{b'} \text{ and } \frac{a}{c} = \frac{a'}{c'} \text{ and } \frac{b}{c} = \frac{b'}{c'}$$

The two approaches express the same thing. Consider for example $\dfrac{a'}{a} = \dfrac{b'}{b}$. If we clear the denominators by multiplying both sides by ab, then we get $a'b = ab'$. If we start with $\dfrac{a}{b} = \dfrac{a'}{b'}$ and clear the denominators, we also get $ab' = a'b$. Consequently, we are free to express similarity in either way. We often select the form that will lead to easier computation.

In case of quadrilaterals, proportional sides alone or equal angles alone can not guarantee similarity.



The square and the rectangle clearly have the same angles and yet the two are not similar. Any enlargement of a square must be a square.



The picture shows a square and a parallelogram with all four of its sides of the same length. Such a quadrilateral is called a rhombus. The two are also not similar although they have the same side lengths.

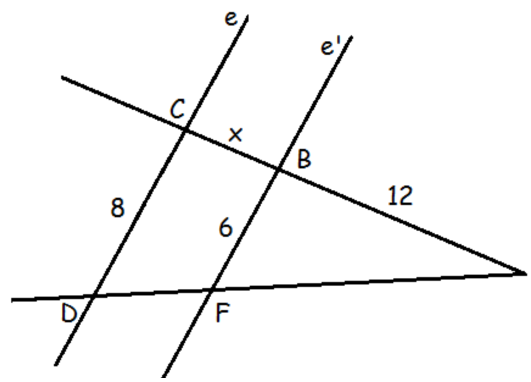
Triangles are simpler than quadrilaterals and so fewer properties are needed to establish similarity. For example, if two triangles have the same angles, then they are similar. There are three cases of similarity.

Theorem: Two triangles are similar to each other if

1. they have the same angles, or
2. if all three pairs of sides are proportional, or
3. if two pairs of sides are proportional and the angle enclosed by them are equal.

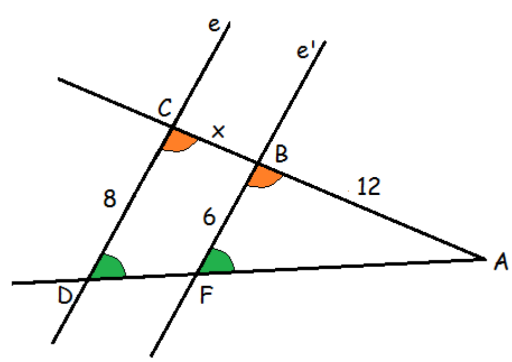
We often establish similarity using angles, and then solve for missing sides using proportionality.

Example 1. Find the value of x based on the figure, given that lines e and e' are parallel.



Solution: First we will use angles to establish that triangles ACD and ABF are similar. Then we use proportonality of sides to find the value of x. Angles ABF and ACD, (marked orange) have equal measures because lines e and e' are parallel. Similarly, angles AFB and ADC, (marked green) have equal measures because lines e and e' are parallel.

The third angle, at point A is shared by the two triangles. (Notice however, that we have similarity with only two pairs of equal angles.) So the two triangles are similar.



Because they are similar, corresponding sides of the two triangles are proportional. We can use this fact to find the value of x. Notice that x, or line segment BC is not a side of any triangles.

It is recommended that we keep track of sides by the angles opposite them. Our eyes seem to work much better with angles than with sides. So we will write the ratio

$$\frac{\text{side opposite the green angle}}{\text{side opposite the angle at point } A}$$

in both triangles. By similarity, this ratio must remain the same.

In triangle ACD, the ratio $\dfrac{\text{side opposite the green angle}}{\text{side opposite the angle at point } A} = \dfrac{x+12}{8}$.

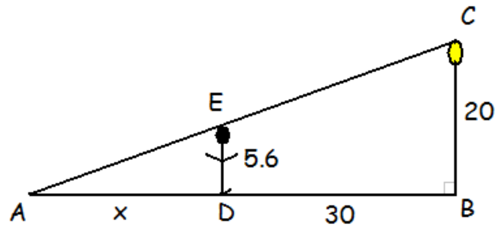
The same ratio in triangle ABF is $\dfrac{12}{6} = 2$. Therefore, we have a linear equation for x.

$$\begin{aligned} \frac{x+12}{8} &= 2 && \text{multiply by } 8 \\ x+12 &= 16 && \text{subtract } 12 \\ x &= 4 \end{aligned}$$

Now that we know the value of x, we see how the proportionality works: 12 to 6 is the same as 16 to 8. Therefore, our solution, $\boxed{x = 4}$ is correct.

Example 2. A 5.6 ft tall person is standing 30 ft away from a street light that is 20 ft tall. How long is her shadow?

Solution: The shadow is created as light is absorbed by the person, so it can not hit the ground behind her. After we draw a picture, we notice that triangles ABC and ADE are similar. They share an angle at point A, and they both also have a right angle.



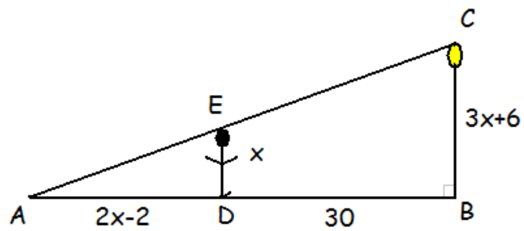
Because the triangles are similar, corresponding sides are proportional. We can write the ratio $\dfrac{\text{horizontal side}}{\text{vertical side}}$ for both triangles. We need to be careful with the horizontal side of the larger triangle: it is not 30 but rather $x+30$.

$$\begin{aligned}
\frac{x}{5.6} &= \frac{x+30}{20} && \text{mulitply by } 20\cdot 5.6\\
20x &= 5.6\,(x+30) && \text{multiply by 10 (to get rid of the decimal)}\\
200x &= 56\,(x+30) && \text{distribute 56}\\
200x &= 56x+1680 && \text{subtract } 56x\\
144x &= 1680 && \text{divide by 144}\\
x &= \frac{1680}{144}=\frac{35}{3}=11\frac{2}{3}
\end{aligned}$$

Thus her shadow is $\boxed{11\frac{2}{3}\text{ ft}}$ long.

Example 3. A 5.6 ft tall person is standing 30 ft away from a street light that is six feet taller than threee times the height of the person. The length of his shadow is two feet less than twice his height. How tall is he?

Solution: The shadow is created as light is absorbed by the person, so it can not hit the ground behind her. After we draw a picture, we notice that triangles ABC and ADE are similar. They share an angle at point A, and they both also have a right angle.



If we label the person's height by x, then the height of the light is $3x+6$ and the length of his shadow is $2x-2$. Now we can use poportionality sides. We can write the ratio $\dfrac{\text{vertical side}}{\text{horizontal side}}$ for both triangles.

$$\begin{aligned}
\frac{x}{2x-2} &= \frac{3x+6}{30+(2x-2)}\\
\frac{x}{2x-2} &= \frac{3x+6}{2x+28} && \text{multiply by } (2x-2)(2x+28)\\
x\,(2x+28) &= (2x-2)(3x+6) && \text{expand products}\\
2x^2+28x &= 6x^2+12x-6x-12 && \text{combine like terms}\\
2x^2+28x &= 6x^2+6x-12 && \text{subtract } 2x^2\text{, subtract } 28x\\
0 &= 4x^2-22x-12 && \text{divide by 2}\\
2x^2-11x-6 &= 0
\end{aligned}$$

Since this is a quadratic equation, we need to factor and apply the zero product rule. We will complete the square.

$$\begin{aligned}
2x^2 - 11x - 6 &= 0 && \text{factor out } 2 \\
2\left(x^2 - \frac{11}{2}x - 3\right) &= 0 && \left(x - \frac{11}{4}\right)^2 = x^2 - \frac{11}{2}x + \frac{121}{16} \\
2\left(x^2 - \frac{11}{2}x + \frac{121}{16} - \frac{121}{16} - \frac{3}{1}\cdot\frac{16}{16}\right) &= 0 \\
2\left(\left(x - \frac{11}{4}\right)^2 - \frac{121}{16} - \frac{48}{16}\right) &= 0 \\
2\left(\left(x - \frac{11}{4}\right)^2 - \frac{169}{16}\right) &= 0 && \sqrt{169} = 13 \\
2\left(\left(x - \frac{11}{4}\right)^2 - \left(\frac{13}{4}\right)^2\right) &= 0 \\
2\left(x - \frac{11}{4} + \frac{13}{4}\right)\left(x - \frac{11}{4} - \frac{13}{4}\right) &= 0 \\
2\left(x + \frac{2}{4}\right)\left(x - \frac{24}{4}\right) &= 0 \\
2\left(x + \frac{1}{2}\right)(x - 6) &= 0 \implies x_1 = -\frac{1}{2} \text{ and } x_2 = 6
\end{aligned}$$

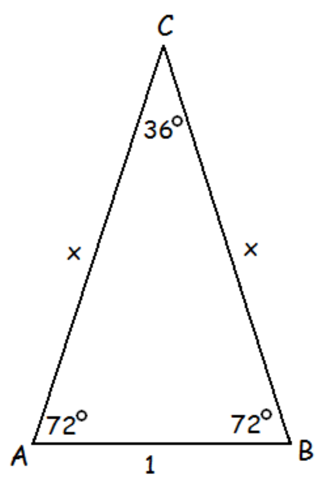
Since distances are never negative, $-\frac{1}{2}$ is easily ruled out. Thus he is $\boxed{6\,\text{ft}}$ tall.

Enrichment

Consider triangle ABC shown on the picture.

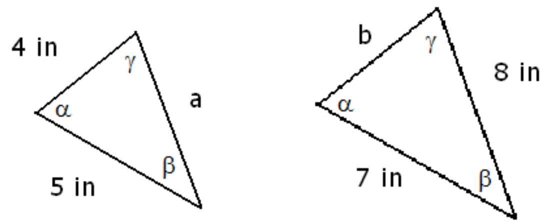
1. Bisect the angle at point A. (To bisect an angle means to split it into two equal angles. Such a line is called the angle bisector) This angle bisector intersects side BC in point D. Prove that triangles ABC and ABD are similar.

2. Use the similarity established in the previous problem to find the exact value of x.

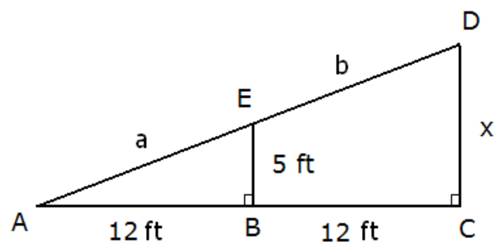


Sample Problems

1. The triangles shown below are similar. Find the exact values of a and b shown on the picture below.



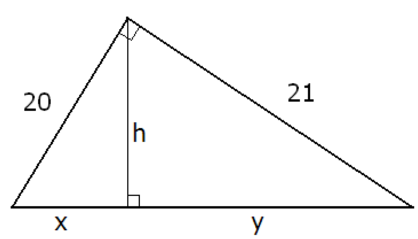
2. Consider the picture shown.
   a) Use the Pythagorean Theorem to find the value of a.
   b) Prove that the triangles ABE and ACD are similar.
   c) Use similar triangles to find the value of x.
   d) Find the value of b.



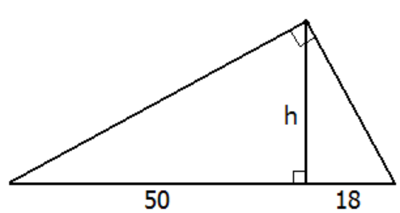
3. a) A person is standing $40\,\mathrm{ft}$ away from a street light that is $30\,\mathrm{ft}$ tall. How tall is he if his shadow is $10\,\mathrm{ft}$ long?
   b) A $6\,\mathrm{ft}$ tall person is standing $24\,\mathrm{ft}$ away from a street light that is $15\,\mathrm{ft}$ tall. How long is her shadow?

4. Prove the following statement. Let ABC be any right triangle, the right angle at point C. The altitude drawn from C to the hypotenuse splits the triangle into two right triangles that are similar to each other and to the original triangle.

5. Find x, y, and h based on the picture below.

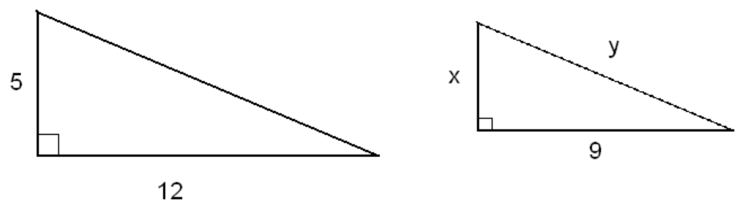


6. The picture below shows a right triangle. Find the length of h, the height drawn to the hypotenuse.

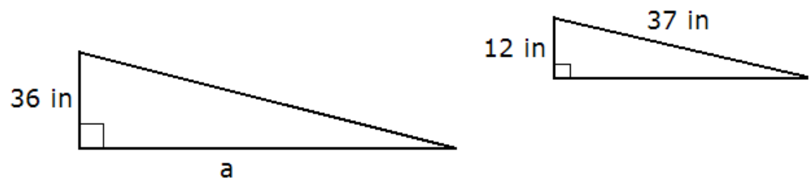


Practice Problems

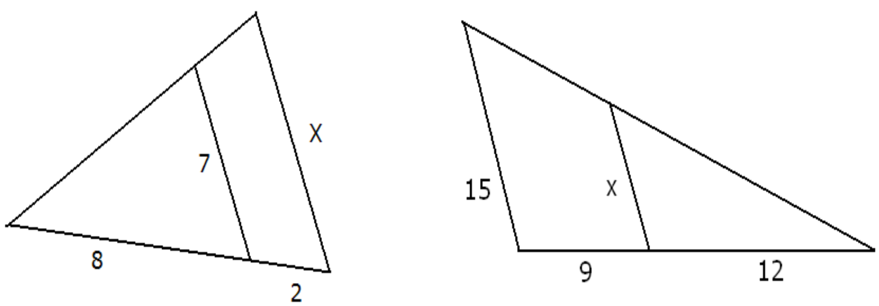
1. The picture below shows two similar right triangles. Find the exact values of x and y.



2. The picture below shows two similar right triangles. Find the exact value of a.



3. Find the value of x based on the figures below.

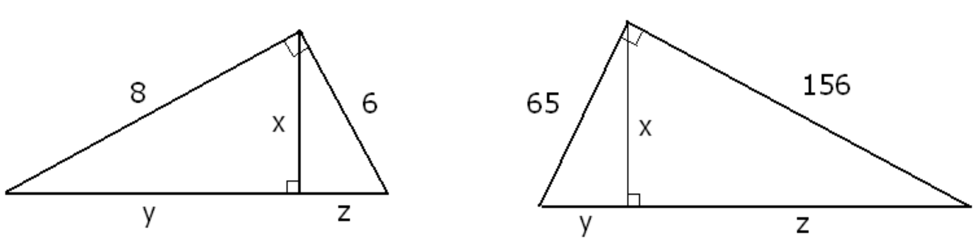


(a) (b)

4. a) A person is standing $24\,\mathrm{ft}$ away from a street light that is $25\,\mathrm{ft}$ tall. How tall is he if his shadow is $6\,\mathrm{ft}$ long?

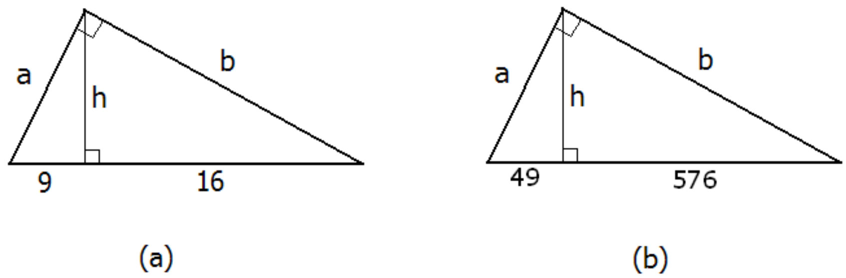
   b) A $5.2\,\mathrm{ft}$ tall person is standing $20\,\mathrm{ft}$ away from a street light that is $15.6\,\mathrm{ft}$ tall. How long is her shadow?

5. Find the exact value of x, y, and z, based on the figures shown below.



(a) (b)

6. Find the exact value of a, b, and h, based on the picture shown below.



Sample Problems - Answers

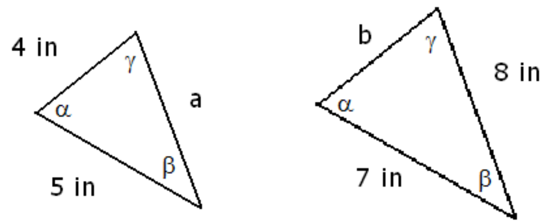
1. $a = \dfrac{40}{7}$ in, $b = \dfrac{28}{5}$ in
2. a) 13 ft b) see solutions c) 10 ft d) 13 ft
3. a) 6 ft b) 16 ft
4. see solutions
5. $h = \dfrac{420}{29}$, $x = \dfrac{400}{29}$, $y = \dfrac{441}{29}$
6. 30 units

Practice Problems - Answers

1. $x = \dfrac{15}{4} = 3.75$, $y = \dfrac{39}{4} = 9.75$
2. 105 in
3. a) $\dfrac{35}{4} = 8.75$ b) $\dfrac{60}{7}$
4. a) 5 ft b) 10 ft
5. a) $x = \dfrac{24}{5} = 4.8$, $y = \dfrac{32}{5} = 6.4$, $z = \dfrac{18}{5} = 3.6$ b) $x = 60$, $y = 25$, $z = 144$
6. a) $h = 12$, $a = 15$, $b = 20$ b) $h = 168$, $a = 175$, $b = 600$

Sample Problems - Solutions

1. The triangles shown below are similar. Find the exact values of a and b shown on the picture below.



Solution: In similar triangles, the ratios of corresponding sides are preserved. To find a, we write the ratio $\dfrac{\text{side opposite angle } \alpha}{\text{side opposite angle } \gamma}$ for both triangles.

$$\frac{\text{side opposite angle } \alpha}{\text{side opposite angle } \gamma} = \frac{a}{5} = \frac{8}{7}$$

We now solve the equation for a.

$$\begin{aligned} \frac{a}{5} &= \frac{8}{7} && \text{multiply both sides by 35} \\ 7a &= 40 && \text{divide by 7} \\ a &= \frac{40}{7} \end{aligned}$$

Similarly, we can find b by writing the ratio $\dfrac{\text{side opposite angle } \beta}{\text{side opposite angle } \gamma}$ for both triangles.

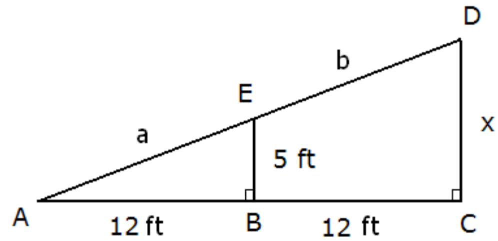
$$\frac{\text{side opposite angle } \beta}{\text{side opposite angle } \gamma} = \frac{4}{5} = \frac{b}{7}$$

We now solve the equation for b.

$$\begin{aligned} \frac{4}{5} &= \frac{b}{7} && \text{multiply both sides by 35} \\ 28 &= 5b && \text{divide by 5} \\ b &= \frac{28}{5} \end{aligned}$$

Thus $a = \dfrac{40}{7}$ in and $b = \dfrac{28}{5}$ in.

2. Consider the picture shown below.



a) Use the Pythagorean Theorem to find the value of a.
Solution: The shorter sides are $5\,\mathrm{ft}$ and $12\,\mathrm{ft}$ long. The hypotenuse is a. We state the Pythagorean Theorem for this triangle and solve the equation for a.

$$\begin{aligned} 5^2+12^2 &= a^2 \\ 25+144 &= a^2 \\ 169 &= a^2 \\ \pm 13 &= a \end{aligned}$$

Since distances can never be negative, $a=-13$ is ruled out. Thus $a=13\,\mathrm{ft}$.

b) Prove that the triangles ABE and ACD are similar.
Solution: First, angles ABE and ACD are both right angles. Second, the two triangles literally share angle EAB (or angle DAC). Finally, if two triangles agree in the measure of two of their angles, the third angles must be equal since in every triangle, the three angles add up to $180°$. The two triangles are similar because they have identical angles.

c) Use similar triangles to find the value of x.
Solution: The triangles $\triangle ABE$ and $\triangle ACD$ are similar. To find x, we write the ratio $\dfrac{\text{side opposite point } A}{\text{horizontal side}}$ for both triangles.

$$\frac{\text{side opposite point } A}{\text{horizontal side}}=\frac{5}{12}=\frac{x}{24}$$

and solve the equation for x.

$$\begin{aligned} \frac{5}{12} &= \frac{x}{24} && \text{multiply both sides by } 24 \\ 10 &= x \end{aligned}$$

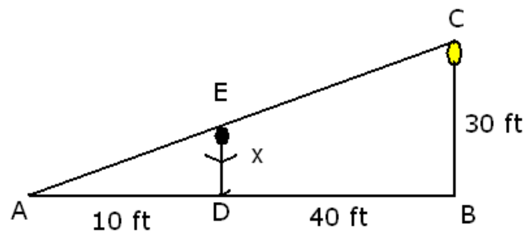
Thus x is $10\,\mathrm{ft}$. Indeed, once we established that the triangles are similar, and noticed that the horizontal side was doubled from $12\,\mathrm{ft}$ to $24\,\mathrm{ft}$, we could easily predict this answer.

d) Find the value of b.
Solution: We can either use similar triangles or the Pythagorean Theorem to find the side AD. Either way, we easily get that $26\,\mathrm{ft}$. However, the length of side AD is not b, but $a+b$. From part a), we know that $a=13\,\mathrm{ft}$.

$$\begin{aligned} 13+b &= 26 \\ b &= 13 \end{aligned}$$

Thus $b=13\,\mathrm{ft}$.

3. a) A person is standing $40\,\text{ft}$ away from a street light that is $30\,\text{ft}$ tall. How tall is he if his shadow is $10\,\text{ft}$ long?
Solution: After we draw a picture, we see that this problem is very similar to the previous one.



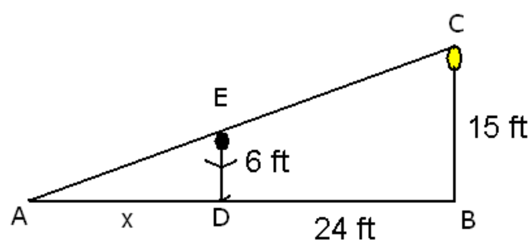
Triangles $\triangle ADE$ and $\triangle ABC$ are similar. We use the ratio $\dfrac{DE}{AD} = \dfrac{BC}{AB}$ and solve for x.

$$\begin{aligned} \frac{x}{10} &= \frac{30}{50} && \text{multiply both sides by } 50 \\ 5x &= 30 && \text{divide by } 5 \\ x &= 6 \end{aligned}$$

Thus the person is $6\,\text{ft}$ tall. Notice that the number 40 did not occur in the equation. It is a common error to use 40 instead of 50.

b) A $6\,\text{ft}$ tall person is standing $24\,\text{ft}$ away from a street light that is $15\,\text{ft}$ tall. How long is her shadow?

Solution: After we draw a picture, write an equation expressing that triangles ADE and ABC are similar.



We can use the same ratio as before, $\dfrac{DE}{AD} = \dfrac{BC}{AB}$ and solve for x.

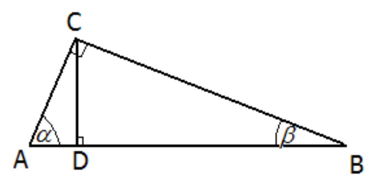
$$\begin{aligned} \frac{6}{x} &= \frac{15}{x+24} && \text{multiply both sides by } x\,(x+24) \\ 6\,(x+24) &= 15x && \text{distribute} \\ 6x+144 &= 15x && \text{subtract } 6x \\ 144 &= 9x && \text{divide by } 9 \\ 16 &= x \end{aligned}$$

Thus her shadow is $16\,\text{ft}$ long.

Note: If the first step, multiplying by $x\left(x+24\right)$ (same as cross-multiplying) is confusing, here is the break-down:

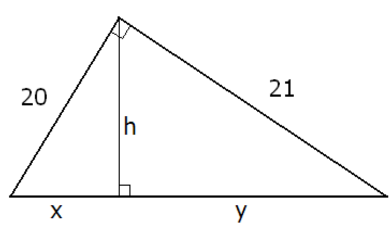
$$\begin{aligned}
\frac{6}{x} &= \frac{15}{x+24} && \text{multiply by } x\left(x+24\right)\\
x\left(x+24\right)\frac{6}{x} &= \frac{15}{x+24}\cdot x\left(x+24\right) && \text{expressing everything as a fraction}\\
\frac{x\left(x+24\right)}{1}\cdot\frac{6}{x} &= \frac{15}{x+24}\cdot\frac{x\left(x+24\right)}{1} &&\\
\frac{x\left(x+24\right)6}{x} &= \frac{15x\left(x+24\right)}{x+24} && \text{cancel}\\
\frac{\left(x+24\right)6}{1} &= \frac{15x}{1} && \text{simplify}\\
\left(x+24\right)6 &= 15x &&
\end{aligned}$$

4. Prove the following statement. Let ABC be any right triangle, the right angle at point C. The altitude drawn from C to the hypotenuse splits the triangle into two right triangles that are similar to each other and to the original triangle.
Solution: Let us draw a picture and use standard labeling of points.

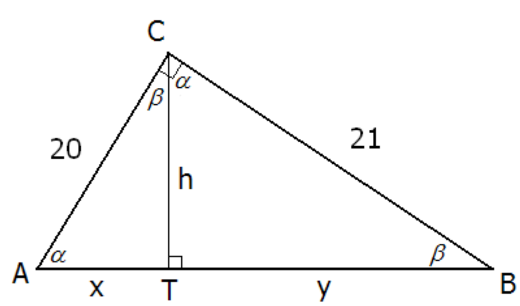


The two triangles created, $\triangle\ ADC$ and $\triangle\ DBC$ are both right triangles. $\triangle ADC$ is similar to the original triangle, because they agree in two angles: the right angle and α. $\triangle DBC$ is similar to the original triangle, because they agree in two angles: the right angle and β. Thus all three triangles are similar. Also, this will be very useful later: $\angle ACD=\beta$ and $\angle BCD=\alpha$.

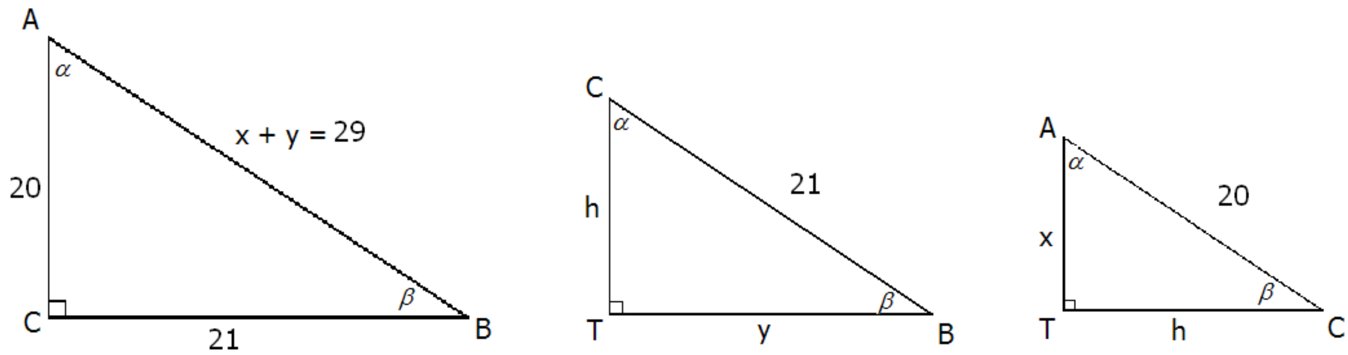
5. Find x, y, and h based on the picture below.



Solution: We can easily find the hypotenuse of this triangle via the Pythagorean Theorem. The hypotenuse, $x+y$ is $\sqrt{20^2+21^2}=29$ units long. Next, let us first label the points, angles and sides in the triangle.



We now re-draw the three similar triangles in a separate figure, all three of them rotated and reflected into the same direction. This way, it is easy to realize what sides correspond to each other. (Hint: start with the anges, they are in the same location. Then identify the points, and finally the sides.)



We can find y using the following ratio in the first two triangles

$$\frac{\text{side opposite } \alpha}{\text{hypotenuse}} = \frac{21}{29} = \frac{y}{21}$$

$$\begin{aligned} \frac{21}{29} &= \frac{y}{21} && \text{multiply both sides by } 21 \cdot 29 \\ 441 &= 29y && \text{divide by 29} \\ \frac{441}{29} &= y \end{aligned}$$

A different ratio in the same triangles can be used to obtain

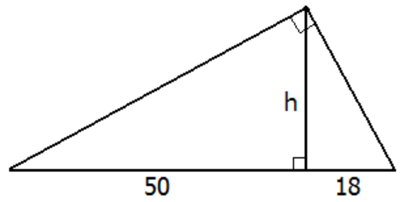
$$\frac{\text{side opposite } \beta}{\text{hypotenuse}} = \frac{20}{29} = \frac{h}{21}$$

$$\begin{aligned} \frac{20}{29} &= \frac{h}{21} && \text{multiply both sides by } 21 \cdot 29 \\ 420 &= 29h && \text{divide by 29} \\ \frac{420}{29} &= h \end{aligned}$$

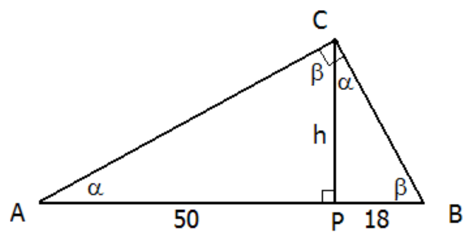
For x, we can simply use the fact that $x + y = 29$ and we already computed $y = \dfrac{441}{29}$.

$$\begin{aligned} x + \frac{441}{29} &= 29 \\ x &= 29 - \frac{441}{29} = \frac{400}{29} \end{aligned}$$

6. The picture below shows a right triangle. Find the length of h, the height drawn to the hypotenuse.



Solution: Let us first label the points, angles and sides in the triangle. As we proved it in the previous problem, the two new triangles are similar to the original triangle.



Consider now the ratio $\dfrac{\text{side opposite } \beta}{\text{side opposite } \alpha}$ in triangles $\triangle APC$ and $\triangle\ PBC$. Since these triangles are similar, this ratio is preserved.

$$\frac{\text{side opposite } \beta}{\text{side opposite } \alpha} = \frac{50}{h} = \frac{h}{18}$$

We solve this equation for h.

$$\begin{aligned} \frac{50}{h} &= \frac{h}{18} \\ 50 \cdot 18 &= h^2 \\ 900 &= h^2 \\ h &= \pm 30 \end{aligned}$$

$h = -30$ is ruled out since distances can not be negative. Thus $h = 30$.

For more documents like this, visit our page at http://www.teaching.martahidegkuti.com and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.