Proof by Contradiction

A very powerful proving technique is what we call **indirect proof**, or **proof by contradiction**.

The logic behind this proving technique is as follows. If we start with a true statement and arrive to other statements by using logically correct steps, these new statements must all be true. Consequently, if start with a statement and use logically correct steps to make conclude other statements, and we arrive to an obviously false statement, we must have started with a false statement. True statements only imply true statements. If our conclusion is false, we must have started with a false statement.

Suppose we want to prove a statement to be true. In case of a proof by contradiction, we formulate the exact opposite of our statement, and, using logically correct steps, we derive an obviously false statement, the contradiction. This proves that the starting staement is false, making its opposite, our statement true.

The fact that $\sqrt{2}$ is irrational can be proven by contradiction.

Definition: A number is **rational** if it can be written as a fraction of two integers.

Definition: A number is **irrational** if it is not rational, i.e. it can not be written as a fraction of two integers.

Theorem: $\sqrt{2}$ is an irrational number.

Proof. Suppose, for a contradiction, that $\sqrt{2}$ is rational, i.e. there exist two integers, a and b $(b \neq 0)$ such that

$$\sqrt{2} = \frac{a}{b}$$

We may also assume that the fraction $\frac{a}{b}$ is in lowest terms, otherwise we could reduce the fraction $\frac{a}{b}$ and replace it with the reduced equivalent. So, let us assume that $\frac{a}{b}$ is in lowest terms, which means that a and b do not share any divisor larger than 1. Now let us square both sides.

$$2 = \frac{a^2}{b^2}$$

Let us multiply both sides by b^2.

$$2b^2 = a^2$$

Since a^2 is twice another integer, it is even. This means that a itself must be even. Let us re-write $a = 2k$ where k is some integer.

$$\begin{aligned} 2b^2 &= (2k)^2 \\ 2b^2 &= 4k^2 \end{aligned}$$

Let us divide both sides by 2. Then we have

$$b^2 = 2k^2$$

Since b^2 is twice another integer, it is even. This means that b itself must be even. We are now done, since we have a contradiction. The following statements do not make sense together.

1. a and b are two integers that do not share any divisors.
2. a is even.
3. b is even

This completes our proof. ■

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