The set of all complex numbers, (denoted by $\mathbb{C}$) is a set containing the set of all real numbers, denoted by $\mathbb{R}$. It is best to think of complex numbers as an extension or enlargement of the set of real numbers. The definitions that established the set of all complex numbers can be explained by this fact. Let us recall the expansion principle.

Definition: The **Expansion Principle** is our desire to protect already true statements whenever new concepts are defined.

If a new definition would result in an already true statement becoming false, mathematicians do not accept the definition.

We will see that most definitions about complex numbers can be explained as our only choice for creating a structure around the real numbers that still has most properties and theorems of the real numbers.

Definition: A complex number is $z = a + bi$ where a and b are real numbers.

Complex numbers are often denoted by z so that we can use x and y to denote its real and imaginary components. For example, we will often see $z = x + yi$ where x and y are real numbers. If y happens to be zero, then $x + yi$ is a real number. Thus, every real number is also a complex number. The set of all real numbers (denoted by $\mathbb{R}$) is a subset of the set of all complex numbers (denoted by $\mathbb{C}$). In short, $\mathbb{R} \subseteq \mathbb{C}$. As a matter of fact, stepping out to the complex numbers is in various ways the last step in our story of expanding our number system. This story is

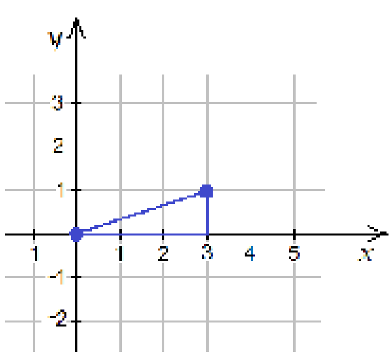
$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

Real numbers are depicted as points on a line, also called the number line, complex numbers are often depicted as points on a plane, also called the complex plane. For example, the complex number $a + bi$ can be depicted as the point (a, b) on the complex plane. This means that the real number 5 can be depicted as $(5, 0)$ on the complex plane, and the number i can be depicted as $(0, 1)$. Notice that this correspondence between numbers and points is a one-to-one correspondence: to each complex number, there exists a unique point in the plane, and to each point in the plane, there exists a unique complex number. The origin represents the real number 0.

Recall the definition of absolute value. The absolute value of a real number is its distance from zero on the number line. As it turns out, the absolute value (or modulus) of a complex number can be defined very similarly.

Definition: The absolute value (or modulus) of a complex number is its distance from zero on the number plane.

For example, the absolute value of $3 + i$, denoted by $|3 + i|$ can is the distance between the points corresponding to $3 + i$ and 0. That is, the distance between $(0, 0)$ and $(3, 1)$. We can use the distance formula or the Pythagorean Theorem.



$$|3 + i| = \sqrt{3^2 + 1^2} = \sqrt{10}$$

Notice that the absolute value of a complex number is (just like that of a real number) a non-negative real number. Also, this definition clearly does not violate the expansion principle. If we take a real number and compute its absolute value as before, as a real number, we get the same result as we get when computing the absolute value as a complex number.

Theorem: The absolute value (or modulus) of a complex number $z = x + yi$ (where x and y are real numbers) is $|z| = \sqrt{x^2 + y^2}$.

Definition: Addition between complex numbers is defined as follows: if $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, then

$$z_1 + z_2 = x_1 + x_2 + (y_1 + y_2)\, i$$

This definition is not at all surprising: we would like to preserve the basic rules from real numbers that allow us to drop parentheses and combine like terms. With that in mind, this definition was pretty much the only way to go. Notice that this definition of addition guarantees that addition of complex numbers inherit the properties from real numbers: addition is commutative, associative, there is an identity element, zero. Once we define multiplication between a real and a every complex number, we will also establish that every complex number has an opposite.

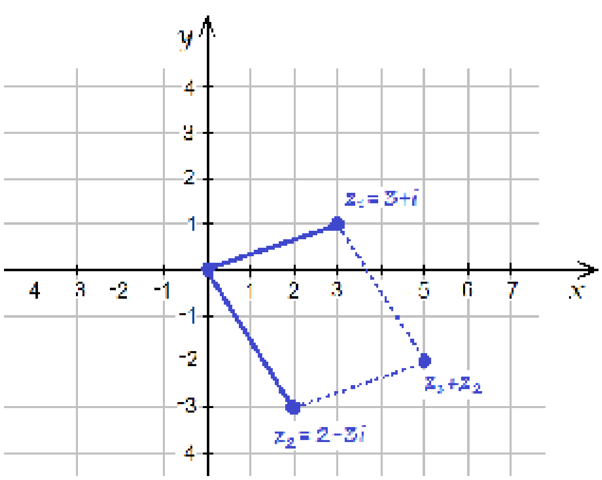
Definition: Multiplication between a real and a complex number: if $z = x + yi$ is a complex number and c is a real number, then

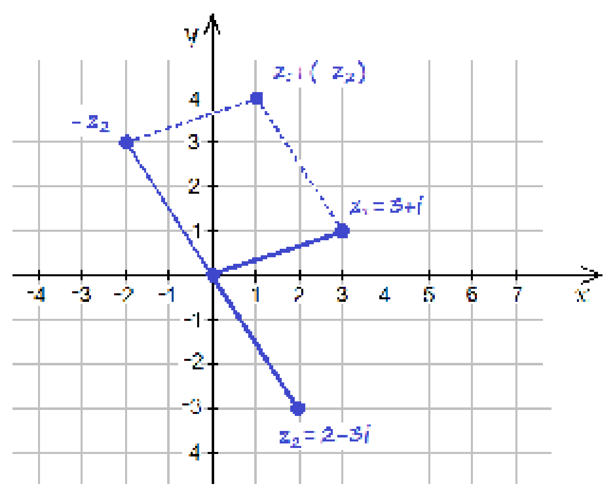
$$cz = c\,(x + yi) = cx + cyi$$

Now we can easily define subtraction between complex numbers. 'To subtract is to add the opposite' will stay true. For example, let $z_1 = 3 + i$ and $z_2 = 2 - 3i$. Then

$$\begin{aligned} z_1 + z_2 &= (3+i) + (2-3i) = 3 + i + 2 - 3i = (3+2) + (i - 3i) = 5 - 2i \qquad \text{and} \\ z_1 - z_2 &= (3+i) - (2-3i) = 3 + i + (-1)\,(2-3i) = 3 + i + (-2) + 3i = (3-2) + (i+3i) = 1 + 4i \end{aligned}$$

Because of the definitions we just discussed, all of these properties have very nice geometric properties. If we depict z_1, and z_2 in the same coordinate system, and connect them to the origin, we obtain a triangle. If we now draw a line through z_1 that is parallel to z_2 and another line through z_2 that is parallel to z_1, we obtain a parallelogram. The fourth vertex of the parallelogram is $z_1 + z_2$. (Later we will see that this behavior is identical to addition of vectors. This is because addition and subtraction of vectors are defined very similarly.)





The next logical step is to define multiplication on complex numbers. It is clear that we would like to preserve properties of multiplication from the real number system: we'd like multiplication to be commutative, associative, we would like to preserve the identity element being 1. And also, we would like to preserve the distributive law. So, using our previous example of z_1 and z_2, we already have an idea about their product.

$$z_1 z_2 = (3+i)(2-3i) = 3 \cdot 2 - 3 \cdot 3i + i \cdot 2 + i(-3i) = 6 - 9i + 2i - 3i^2 = 6 - 7i - 3i^2$$

We will know the product $z_1 z_2$ if we could just figure out what is i^2. At this point of the construction of the complex numbers, we had the freedom to define i^2 any way we liked. Of course, different choices would result in different kind of rules we would observe. How i^2 was defined has a lot to do with the expansion theorem. We will see that if we want a simple rule from the real numbers, $|x||y| = |xy|$ to be true on the complex numbers, then i^2 can only have the value -1.

Theorem: If for all complex numbers z_1 and z_2, the statement $|z_1||z_2| = |z_1 z_2|$ is true, then i^2 can only be defined as -1.

Proof: Assume that the complex number $i^2 = a + bi$. So, we will know the complex number i^2 if we know the real numbers a and b. In what follows, we will obtain two equations in a and b. First, let us look at the statement

$$|i| \cdot |i| = \left|i^2\right|$$

Since $|i| = 1$ and $\left|i^2\right| = |a+bi| = \sqrt{a^2+b^2}$, this equation becomes

$$\begin{aligned} 1 \cdot 1 &= \sqrt{a^2+b^2} \qquad \text{squaring both sides gives us} \\ 1 &= a^2 + b^2 \end{aligned}$$

Next, let us look at the statement

$$|1+i| \cdot |1-i| = |(1+i)(1-i)|$$

On the left-hand side, we can determine the absolute value of each numbers:

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2} \quad \text{and} \quad |1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$$

On the right-hand side, we simplify $(1+i)(1-i)$ as much as we can: $(1+i)(1-i) = 1 - i^2$. So, our equation

$$\begin{aligned} |1+i| \cdot |1-i| &= |(1+i)(1-i)| \quad \text{becomes} \\ \sqrt{2} \cdot \sqrt{2} &= \left|1-i^2\right| \\ 2 &= \left|1-i^2\right| \end{aligned}$$

Recall that $i^2 = a + bi$, where a and b are real numbers.

$$\begin{aligned} 2 &= |1-(a+bi)| \\ 2 &= |1-a-bi| \\ 2 &= |(1-a)-bi| \\ 2 &= \sqrt{(1-a)^2+b^2} \qquad \text{square both sides} \\ 4 &= (1-a)^2+b^2 \end{aligned}$$

So, we have two equations:

$$\begin{aligned} a^2+b^2 &= 1 \\ (1-a)^2+b^2 &= 4 \end{aligned}$$

This should be enough for us to find a and b. Let us work on the second equation:

$$\begin{aligned}
(1-a)^2+b^2 &= 4 \\
1-2a+\underbrace{a^2+b^2} &= 4 \qquad \text{according to the other equation, } a^2+b^2=1 \\
1-2a+1 &= 4 \\
2-2a &= 4 \\
2 &= 4+2a \\
-2 &= 2a \\
-1 &= a
\end{aligned}$$

Now let us go back to the first equation:

$$\begin{aligned}
a^2+b^2 &= 1 \quad \text{and now we know that } a=-1 \\
(-1)^2+b^2 &= 1 \\
b^2 &= 0 \\
b &= 0
\end{aligned}$$

Thus $i^2 = a+bi = -1+0i = -1$. This concludes our proof. ■

What did we prove here? Did we prove that i^2 is -1? No, we have only proved that if we wanted to preserve the rule $|x|\,|y| = |xy|$ from the real numbers, our only option for the value of i^2 is -1. We didn't even prove that if we define i^2 to be -1, some other things wouldn't go wrong. Or even that the rule $|x|\,|y| = |xy|$ would *always* work. We just used this rule in two cases: $|i|\,|i| = \left|i^2\right|$ and $|1+i|\,|1-i| = |(1+i)(1-i)|$. Mathematicians still needed to verify that all the rules we desire to preserve are safe under the definition $i^2=-1$. And, as it turned out, they all (or at least most) are safe.

Definition: $i^2 = -1$

Now we can easily define multiplication on complex numbers. Our unfinished example,

$$z_1z_2 = (3+i)(2-3i) = 6-9i+2i-3i^2 = 6-7i-3(-1) = 6-7i+3 = 9-7i$$

In general, if $z_1 = x_1+y_1i$ and $z_2 = x_2+y_2i$, then

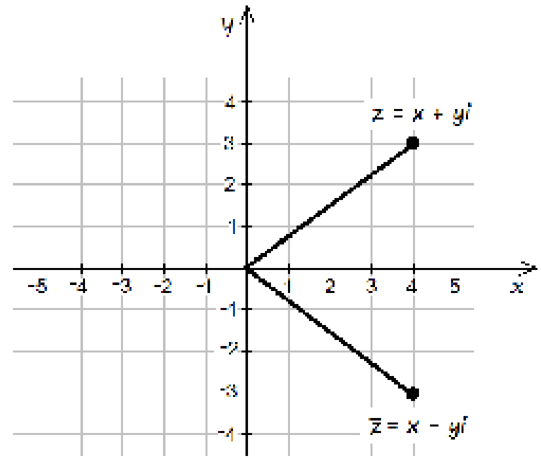
$$\begin{aligned}
z_1z_2 &= (x_1+y_1i)(x_2+y_2i) = x_1x_2+x_1y_2i+x_2y_1i+y_1y_2\left(i^2\right) = x_1x_2+(x_1y_2+x_2y_1)\,i+y_1y_2(-1) \\
&= x_1x_2-y_1y_2+(x_1y_2+x_2y_1)\,i
\end{aligned}$$

Theorem: If $z_1 = x_1+y_1i$ and $z_2 = x_2+y_2i$, then $z_1z_2 = x_1x_2-y_1y_2+(x_1y_2+x_2y_1)\,i$

The last operation that is left undefined is division. But before we do that, we will define the complex conjugate of a number. As we will see later, complex conjugates are exremely useful.

Definition: The complex conjugate of a number z, denoted by $\overline{z}$ is defined as follows. If $z = x+yi$, then $\overline{z} = x-yi$.

A complex number and its complex conjugate are symmetrical to the $x-$axis.



An interesting and important fact: if add or multiply a complex number and its conjugate, the result in both cases is a real number. Consider for example $z = 1 + 7i$. Then

$$z + \overline{z} = (1 + 7i) + (1 - 7i) = 2 \quad \text{and} \quad z\overline{z} = (1 + 7i)(1 - 7i) = 1 - 49i^2 = 1 + 49 = 50$$

Actually, more is true: if we multiply a complex number z and its conjugate $\overline{z}$, the result is the square of the absolute value of z.

Theorem: $z\overline{z} = |z|^2$.

Proof: Supose that $z = x + yi$. Then $\overline{z} = x - yi$ and recall that $|x + yi| = \sqrt{x^2 + y^2}$

$$z\overline{z} = (x + yi)(x - yi) = x^2 - y^2\left(i^2\right) = x^2 + y^2 = |z|^2$$

We are now ready to define division on complex numbers. First, we define division of a complex number by a real number. We actually already have this definition, remember, to divide is to multiply by the reciprocal. So, if c is a real number, not zero, and $z = x + yi$ is any complex number, then

$$\frac{z}{c} = \frac{x + yi}{c} = \frac{1}{c}(x + yi) = \frac{x}{c} + \frac{y}{c}i$$

When we are dividing by a complex number z, we will use the conjugate $\overline{z}$ to transform the division into another one where the denominator is real. Let us look at, for example, the division $\dfrac{3 - 4i}{2 + i}$. To make the denominator real, we multiply both numerator and denominator by the conjugate of $2 + i$. 1 is still the multiplicative identity so multiplication by 1 still does not change the value of any complex number.

$$\frac{3 - 4i}{2 + i} = \frac{3 - 4i}{2 + i} \cdot 1 = \frac{3 - 4i}{2 + i} \cdot \frac{2 - i}{2 - i} = \frac{(3 - 4i)(2 - i)}{(2 + i)(2 - i)} = \frac{6 - 3i - 8i + 4i^2}{4 - 2i + 2i - i^2} = \frac{6 - 11i + 4(-1)}{4 - (-1)} = \frac{2 - 11i}{5}$$

and now this is just a division by a real number:

$$\frac{2 - 11i}{5} = \frac{1}{5}(2 - 11i) = \frac{2}{5} - \frac{11}{5}i$$

Some questions are now different depending on whether we are working within $\mathbb{R}$ or $\mathbb{C}$. For example, $\sqrt{-9}$ is undefined over the real numbers but it is $3i$ over the complex numbers.

Another very important property of complex numbers is that over $\mathbb{C}$, every polynomial can be factored into a product of linear factors. Consider for example the polynomial x^2+1. While it is irreducible over $\mathbb{R}$, it is not over $\mathbb{C}$. We apply the difference of squares theorem:

$$x^2+1 = x^2-(-1) = x^2-i^2 = (x+i)(x-i)$$

We can factor more complicated quadratic expressions by completing the square.

$$\begin{aligned} x^2-6x+13 &= \underbrace{x^2-6x+9}-9+13 = (x-3)^2+4 = (x-3)^2-(-4) \\ &= (x-3)^2-(2i)^2 = (x-3+2i)(x-3-2i) \end{aligned}$$

So, every quadratic equation has solution(s) over the complex numbers.

While we preserved most of the mathematics we had over the real numbers, we did lose some parts. For example, we cannot order complex numbers. In other words, $<$ and $\leq$ are meaningless over the complax numbers, unless they happen to be real.

Another rule we have lost is that $\sqrt{ab}=\sqrt{a}\sqrt{b}$. While this was true among real numbers, it is no longer ture over the complex numbers. Here is an easy example to consider:

$$\sqrt{-4}\sqrt{-4} = (2i)(2i) = 4i^2 = -4 \quad \text{but} \quad \sqrt{-4(-4)} = \sqrt{16} = 4$$

So we need to be careful not to use this rule over $\mathbb{C}$.

Sample Problems

Perform the indicated operations and simplify.

1. $|3-5i|$
2. $(2-5i)+(1-i)$
3. $3+5i-(2+3i)$
4. $3(2-i)-2(3+i)$
5. $-i(3+i)$
6. $3(2-i)-2i(3+i)$
7. $(2-3i)(5+2i)$
8. $(3-2i)^2$
9. $(3-2i)(3+2i)$
10. $(7-3i)(i+1)$
11. $(3+5i)(-3+5i)$
12. $(1-3i)^2(1+3i)^2$
13. $(1-i)^4$
14. $\dfrac{2i}{1-i}$
15. $\dfrac{1+7i}{3i+1}$
16. $\dfrac{10+5i}{3+4i}$
17. $\dfrac{5}{2-i}$
18. $\dfrac{8i-1}{2i+3}$
19. $\dfrac{5}{1+2i}-\dfrac{10i}{1-2i}$
20. $\dfrac{(3+2i)(5-3i)-(7-2i)(3-i)}{3-4i}$
21. $\dfrac{(3-i)^2-(1+3i)^2}{2i+2}$

Completely factor each of the following over the complex numbers.

22. x^2+9
23. $x^2-10x+29$
24. x^4-1

Solve each of the following equations over the complex numbers.

25. $x^2=4x-29$
26. $x^2+3=2x$

27*. Find z such that $z^2=-21+20i$.

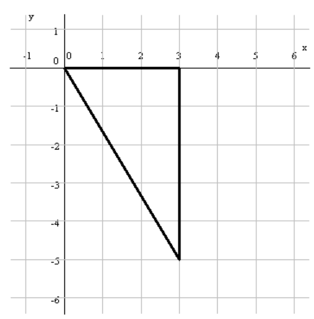
Sample Problems - Answers

1. $\sqrt{34}$ 2. $3-6i$ 3. $1+2i$ 4. $-5i$ 5. $1-3i$ 6. $8-9i$ 7. $16-11i$ 8. $5-12i$ 9. 13
10. $10+4i$ 11. -34 12. 100 13. -4 14. $-1+i$ 15. $\frac{11}{5}+\frac{2}{5}i$ 16. $2-i$ 17. $2+i$
18. $1+2i$ 19. $5-4i$ 20. $-2+2i$ 21. $1-7i$ 22. $(x+3i)(x-3i)$ 23. $(x-5+2i)(x-5-2i)$
24. $(x+1)(x-1)(x+i)(x-i)$ 25. $2+5i$ and $-2-5i$ 26. $1+\sqrt{2}i$ and $1-\sqrt{2}i$ 27. $2+5i$ and $-2-5i$

Sample Problems - Solutions

Perform the indicated operations and simplify.

1. $|3-5i|$
   Solution: The absolute value of complex number is its distance from zero on the number plane. In the case of $3-5i$ this means the distance between the points $(3,-5)$ and $(0,0)$



   We state the Pythagorean theorem on the right triangle and obtain the equation

$$\begin{aligned} 3^2+5^2 &= x^2 \\ 34 &= x^2 \\ \pm\sqrt{34} &= x \end{aligned}$$

   Since distances can never be negative, we rule out the negative solution and so the answer is $\sqrt{34}$.

2. $(2-5i)+(1-i)$
   Solution: We simply drop the parentheses and combine like terms.

$$(2-5i)+(1-i)=2-5i+1-i=2+1+(-5i-i)=3-6i$$

3. $3+5i-(2+3i)$
   Solution: To subtract is to add the opposite. Then we drop the parentheses and combine like terms.

$$3+5i-(2+3i)=3+5i+(-1)(2+3i)=3+5i+(-2-3i)=3-2+5i-3i=1+2i$$

4. $3(2-i)-2(3+i)$
   Solution: We apply the distributive law and combine like terms.
$$3(2-i)-2(3+i)=6-3i-6-2i=-5i$$

5. $-i(3+i)$
   Solution: We apply the distributive law and combine like terms.
$$-i(3+i)=-3i-i^2=-3i-(-1)=1-3i$$

6. $3(2-i)-2i(3+i)$
   Solution: We apply the distributive law and combine like terms.
$$3(2-i)-2i(3+i)=6-3i-6i-2i^2=6-9i-2(-1)=6-9i+2=8-9i$$

7. $(2-3i)(5+2i)$
   Solution: We apply the distributive law (in this case, FOIL) and combine like terms.
$$(2-3i)(5+2i)=10+4i-15i-6i^2=10+4i-15i-6(-1)=16-11i$$

8. $(3-2i)^2$
   Solution: We apply the distributive law (in this case, FOIL) and combine like terms.
$$(3-2i)^2=(3-2i)(3-2i)=9-6i-6i+4i^2=9-6i-6i+4(-1)=5-12i$$

9. $(3-2i)(3+2i)$
   Solution: We apply the distributive law (in this case, FOIL) and combine like terms.
$$(3-2i)(3+2i)=(3-2i)(3+2i)=9-6i+6i-4i^2=9-4(-1)=13$$
   Since we multiplied conjugates, we have the difference of squares theorem.

10. $(7-3i)(i+1)$
   Solution: We apply the distributive law (in this case, FOIL) and combine like terms.
$$(7-3i)(i+1)=7i+7-3i^2-3i=7i+7-3(-1)-3i=10+4i$$

11. $(3+5i)(-3+5i)$
   Solution: We apply the distributive law (in this case, FOIL) and combine like terms. After FOIL, there will be cancellations because we are multiplying a number and its conjugate, so the product is the difference of the two squares.
$$(3+5i)(-3+5i)=-9+15i-15i+25i^2=-9+25(-1)=-9-25=-34$$

12. $(1-3i)^2(1+3i)^2$
   Solution 1: We apply the distributive law for each square and multiply the two numbers we obtained.
$$\begin{aligned}(1-3i)^2(1+3i)^2 &= \left(1-3i-3i+9i^2\right)\left(1+3i+3i+9i^2\right)\\ &= (1-6i+9(-1))(1+6i+9(-1))\\ &= (-8-6i)(-8+6i)=-2(4+3i)\,2(-4+3i)\\ &= -4(4+3i)(-4+3i)=-4\left(-16+12i-12i+9i^2\right)\\ &= -4(-16+9(-1))=-4(-25)=100\end{aligned}$$

Solution 2: We "outsmart" the problem by computing the product in a different order. We use the exponent rule $a^2b^2 = (ab)^2$.

$$\begin{aligned}(1-3i)^2(1+3i)^2 &= ((1-3i)(1+3i))^2 \\ &= \left(1+3i-3i-9i^2\right)^2 \\ &= (1-9(-1))^2 = 10^2 = 100\end{aligned}$$

13. $(1-i)^4$

Solution: We will compute $\left((1-i)^2\right)^2$.

$$\begin{aligned}(1-i)^4 &= \left((1-i)^2\right)^2 = ((1-i)(1-i))^2 = \left(1-i-i+i^2\right)^2 \\ &= (1-2i+(-1))^2 = (-2i)^2 = 4i^2 = 4(-1) = -4\end{aligned}$$

14. $\dfrac{2i}{1-i}$

Solution: We will use the same technique as with rationalizing quotients with irrational denominator: we multiply the fraction by 1, where both numerator and denominator are the conjugate of the denominator.

$$\begin{aligned}\frac{2i}{1-i} &= \frac{2i}{1-i}\cdot 1 = \frac{2i}{1-i}\cdot\frac{1+i}{1+i} = \frac{2i(1+i)}{1^2-i^2} = \frac{2i+2i^2}{1-(-1)} = \frac{2i+2(-1)}{2} = \frac{-2+2i}{2} \\ &= \frac{2(-1+i)}{2} = -1+i\end{aligned}$$

15. $\dfrac{1+7i}{3i+1}$

Solution: We first multiply the fraction by 1, where both numerator and denominator are the conjugate of the denominator.

$$\begin{aligned}\frac{1+7i}{3i+1} &= \frac{1+7i}{1+3i}\cdot 1 = \frac{1+7i}{1+3i}\cdot\frac{1-3i}{1-3i} = \frac{(1+7i)(1-3i)}{1^2-(3i)^2} = \frac{1-3i+7i-21i^2}{1-9i^2} = \frac{1+4i-21(-1)}{1-9(-1)} \\ &= \frac{1+4i+21}{10} = \frac{22+4i}{10} = \frac{22}{10}+\frac{4}{10}i = \frac{11}{5}+\frac{2}{5}i\end{aligned}$$

16. $\dfrac{10+5i}{3+4i}$

Solution: We first multiply the fraction by 1, where both numerator and denominator are the conjugate of the denominator.

$$\begin{aligned}\frac{10+5i}{3+4i} &= \frac{10+5i}{3+4i}\cdot 1 = \frac{10+5i}{3+4i}\cdot\frac{3-4i}{3-4i} = \frac{(10+5i)(3-4i)}{(3+4i)(3-4i)} = \frac{30-40i+15i-20i^2}{9-12i+12i-16i^2} \\ &= \frac{30-25i-20(-1)}{9-16(-1)} = \frac{30-25i+20}{9+16} = \frac{50-25i}{25} = \frac{25(2-i)}{25} = 2-i\end{aligned}$$

17. $\dfrac{5}{2-i}$

Solution: We first multiply the fraction by 1, where both numerator and denominator are the conjugate of the denominator.

$$\begin{aligned}\frac{5}{2-i} &= \frac{5}{2-i}\cdot 1 = \frac{5}{2-i}\cdot\frac{2+i}{2+i} = \frac{5(2+i)}{(2+i)(2-i)} = \frac{10+5i}{2^2-i^2} = \frac{10+5i}{4-(-1)} = \frac{10+5i}{4+1} = \frac{10+5i}{5} \\ &= \frac{5(2+i)}{5} = 2+i\end{aligned}$$

18. $\dfrac{8i-1}{2i+3}$

Solution: We will use the same technique as with rationalizing quotients with irrational denominator: we multiply the fraction by 1, where both numerator and denominator are the conjugate of the denominator.

$$\begin{aligned}\frac{8i-1}{2i+3} &= \frac{-1+8i}{3+2i}\cdot 1=\frac{-1+8i}{3+2i}\cdot\frac{3-2i}{3-2i}=\frac{(-1+8i)(3-2i)}{3^2-(2i)^2}=\frac{-3+2i+24i-16i^2}{9-4i^2}\\ &= \frac{-3+26i-16(-1)}{9-4(-1)}=\frac{13+26i}{13}=\frac{13(1+2i)}{13}=1+2i\end{aligned}$$

19. $\dfrac{5}{1+2i}-\dfrac{10i}{1-2i}$

Solution: We perform each division and then subtract. As you see, it is sometimes wise to wait with the distribution on the top because we save work as terms might cancel out before the multiplication.

$$\frac{5}{1+2i}=\frac{5}{1+2i}\cdot 1=\frac{5}{1+2i}\cdot\frac{1-2i}{1-2i}=\frac{5(1-2i)}{1^2-(2i)^2}=\frac{5(1-2i)}{1-4i^2}=\frac{5(1-2i)}{1-4(-1)}=\frac{5(1-2i)}{5}=1-2i$$

$$\begin{aligned}\frac{10i}{1-2i} &= \frac{10i}{1-2i}\cdot 1=\frac{10i}{1-2i}\cdot\frac{1+2i}{1+2i}=\frac{10i(1+2i)}{1^2-(2i)^2}=\frac{10i(1+2i)}{1-4i^2}=\frac{10i(1+2i)}{1-4(-1)}\\ &= \frac{10i(1+2i)}{5}=2i(1+2i)=2i+4i^2=-4+2i\end{aligned}$$

We are ready to subtract:

$$\frac{5}{1+2i}-\frac{10i}{1-2i}=1-2i-(-4+2i)=1-2i+4-2i=5-4i$$

20. $\dfrac{(3+2i)(5-3i)-(7-2i)(3-i)}{3-4i}$

Solution: We apply the order of operations agreement and perform the multiplications first, then the subtraction, and finally the division.

$$(3+2i)(5-3i)=15-9i+10i-6i^2=15+i-6(-1)=21+i$$
$$(7-2i)(3-i)=21-7i-6i+2i^2=21-13i+2(-1)=19-13i$$
$$(3+2i)(5-3i)-(7-2i)(3-i)=(21+i)-(19-13i)=21+i-19+13i=2+14i$$

$$\begin{aligned}\frac{2+14i}{3-4i} &= \frac{2+14i}{3-4i}\cdot 1=\frac{2+14i}{3-4i}\cdot\frac{3+4i}{3+4i}=\frac{6+8i+42i+56i^2}{3^2-(4i)^2}=\frac{6+50i+56(-1)}{9-(-16)}\\ &= \frac{-50+50i}{25}=\frac{25(-2+2i)}{25}=-2+2i\end{aligned}$$

21. $\dfrac{(3-i)^2-(1+3i)^2}{2i+2}$

Solution:

$$\begin{aligned}\frac{(3-i)^2-(1+3i)^2}{2i+2} &= \frac{9-6i+i^2-\left(1+6i+9i^2\right)}{2i+2}=\frac{9-6i+(-1)-(1+6i-9)}{2i+2}\\ &= \frac{8-6i-(-8+6i)}{2i+2}=\frac{8-6i+8-6i}{2i+2}=\frac{16-12i}{2i+2}=\frac{2(8-6i)}{2(1+i)}\\ &= \frac{8-6i}{1+i}\cdot 1=\frac{8-6i}{1+i}\cdot\frac{1-i}{1-i}=\frac{8-8i-6i+6i^2}{1^2-i^2}=\frac{2-14i}{2}\\ &= \frac{2(1-7i)}{2}=1-7i\end{aligned}$$

Completely factor each of the following over the complex numbers.

22. x^2+9

Solution: $x^2+9=x^2-(-9)=x^2-(3i)^2=(x+3i)(x-3i)$

We can check via multiplication: $(x+3i)(x-3i)=x^2-3ix+3ix-9i^2=x^2-(-9)=x^2+9$.

So our solution is correct.

23. $x^2-10x+29$

Solution:

$$\begin{aligned} x^2-10x+29 &= \underbrace{x^2-10x+25}-25+29=(x-5)^2+4=(x-5)^2-(-4)=(x-5)^2-(2i)^2 \\ &= (x-5+2i)(x-5-2i) \end{aligned}$$

We can check via multiplication, but first we will make a slight modification that will make the computation easier. We will re-write $x-5+2i$ as $x-(5-2i)$ and $x-5-2i$ as $x-(5+2i)$.

$$\begin{aligned} (x-5+2i)(x-5-2i) &= (x-(5-2i))(x-(5+2i)) \\ &= x^2-(5-2i)x-(5+2i)x+(5-2i)(5+2i) \\ &= x^2-x(5-2i+5+2i)+25-(2i)^2=x^2-10x+29 \end{aligned}$$

and so our solution is correct.

24. x^4-1

Solution:

$$x^4-1=\left(x^2\right)^2-1^2=\left(x^2+1\right)\left(x^2-1\right)=\left(x^2-(-1)\right)\left(x^2-1\right)=(x+i)(x-i)(x+1)(x-1)$$

Solve each of the following equations over the complex numbers.

25. $x^2=4x-29$

Solution 1 (Completing the square).

$$\begin{aligned} x^2 &= 4x-29 \\ x^2-4x+29 &= 0 && \text{complete the square} \\ \underbrace{x^2-4x+4}-4+29 &= 0 \\ (x-2)^2+25 &= 0 \\ (x-2)^2-(-25) &= 0 \\ (x-2)^2-(5i)^2 &= 0 && \text{factor via the difference of squares theorem} \\ (x-2+5i)(x-2-5i) &= 0 \end{aligned}$$

$$\begin{aligned} x-2+5i &= 0 \qquad \text{or} \qquad x-2-5i=0 \\ x &= 2-5i \qquad \text{or} \qquad x=2+5i \end{aligned}$$

Solution 2 (The quadratic formula)

$$\begin{aligned} x^2 &= 4x-29 \\ x^2-4x+29 &= 0 && a=1,\ b=-4,\text{ and } c=29 \end{aligned}$$

$$x_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 29}}{2 \cdot 1} = \frac{4 \pm \sqrt{16 - 116}}{2} = \frac{4 \pm \sqrt{-100}}{2} = \frac{4 \pm 10i}{2} = 2 \pm 5i$$

We check $x = 2 - 5i$.

$$\begin{aligned} \text{LHS} &= x^2 = (2-5i)^2 = 4 - 20i + 25i^2 = 4 - 20i - 25 = -21 - 20i \\ \text{RHS} &= 4x - 29 = 4(2-5i) - 29 = 8 - 20i - 29 = -21 - 20i \end{aligned}$$

So $x = 2 - 5i$ is a solution. We also check $x = 2 + 5i$.

$$\begin{aligned} \text{LHS} &= x^2 = (2+5i)^2 = 4 + 20i + 25i^2 = 4 + 20i - 25 = -21 + 20i \\ \text{RHS} &= 4x - 29 = 4(2+5i) - 29 = 8 + 20i - 29 = -21 + 20i \end{aligned}$$

So $x = 2 + 5i$ is also a solution.

26. $x^2 + 3 = 2x$

Solution 1 (Completing the square).

$$\begin{aligned} x^2 + 3 &= 2x \\ x^2 - 2x + 3 &= 0 && \text{complete the square} \\ \underbrace{x^2 - 2x + 1} - 1 + 3 &= 0 \\ (x-1)^2 + 2 &= 0 \\ (x-1)^2 - (-2) &= 0 \\ (x-1)^2 - \left(\sqrt{2}i\right)^2 &= 0 && \text{factor via the difference of squares theorem} \\ \left(x - 1 + \sqrt{2}i\right)\left(x - 1 - \sqrt{2}i\right) &= 0 \end{aligned}$$

$$\begin{aligned} x - 1 + \sqrt{2}i &= 0 \qquad \text{or} \qquad x - 1 - \sqrt{2}i = 0 \\ x &= 1 - \sqrt{2}i \qquad \text{or} \qquad x = 1 + \sqrt{2}i \end{aligned}$$

Solution 2 (The quadratic formula)

$$\begin{aligned} x^2 + 3 &= 2x \\ x^2 - 2x + 3 &= 0 && a = 1,\ b = -2, \text{ and } c = 3 \end{aligned}$$

$$x_{1,2} = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{2 \pm \sqrt{4 - 12}}{2} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm \sqrt{8}i}{2} = \frac{2 \pm 2\sqrt{2}i}{2} = 1 \pm \sqrt{2}i$$

We check $x = 1 + \sqrt{2}i$.

$$\begin{aligned} \text{LHS} &= x^2 + 3 = \left(1 + \sqrt{2}i\right)^2 + 3 = 1 + 2i^2 + 2\sqrt{2}i + 3 = 2 + 2\sqrt{2}i \\ \text{RHS} &= 2x = 2\left(1 + \sqrt{2}i\right) = 2 + 2\sqrt{2}i \end{aligned}$$

So $x = 1 + \sqrt{2}i$ is a solution. We also check $x = 1 - \sqrt{2}i$.

$$\begin{aligned} \text{LHS} &= x^2 + 3 = \left(1 - \sqrt{2}i\right)^2 + 3 = 1 + 2i^2 - 2\sqrt{2}i + 3 = 2 - 2\sqrt{2}i \\ \text{RHS} &= 2x = 2\left(1 - \sqrt{2}i\right) = 2 - 2\sqrt{2}i \end{aligned}$$

So $x = 1 - \sqrt{2}i$ is also a solution.

27. Find z such that $z^2 = -21 + 20i$.
Solution: Let $z = x + yi$. (Remember, x and y are real numbers.) Then

$$z^2 = (x + yi)^2 = x^2 + 2x(yi) + (yi)^2 = x^2 + 2xyi + y^2i^2 = \underbrace{x^2 - y^2}_{\text{real part}} + \underbrace{2xyi}_{\text{imaginary part}}$$

$$\begin{aligned} z^2 &= -21 + 10i \\ \left(x^2 - y^2\right) + (2xy)\, i &= -21 + 20i \end{aligned}$$

The equation of complex numbers gives us a system of equations on real numbers.

$$\begin{aligned} x^2 - y^2 &= -21 \\ 2xy &= 20 \qquad \Longrightarrow \qquad xy = 10 \end{aligned}$$

We square the second equation

$$\begin{aligned} x^2 + 21 &= y^2 \\ x^2y^2 &= 100 \end{aligned}$$

We solve the system by substitution:

$$\begin{aligned} x^2\left(x^2 + 21\right) &= 100 \\ x^4 + 21x^2 - 100 &= 0 \\ \left(x^2 + 25\right)\left(x^2 - 4\right) &= 0 \end{aligned}$$

$$x_1^2 = -25 \qquad x_2^2 = 4$$

Since the square of real numbers can not be negative, only $x^2 = 4$ can apply to this problem. If $x = 2$, we get $y = 5$ and so $z = 2 + 5i$. If $x = -2$, then $y = -5$ and so $z = -2 - 5i$. This does make sense: a number and its opposite will square to the same thing even among complex numbers: $(-z)^2 = (-z)(-z) = z^2$.

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