Sample Problems

1. Find an equation for the circle centered at $(2,-1)$ with radius $r=5$ units.
2. Graph the equation $10x+x^2+y^2=6(y-5)$.
3. Consider the circle $(x-1)^2+(y+3)^2=20$. Find all points on the circle with $y-$coordinate being -5.
4. Find the coordinates of all points where the circle $x^2+(y-3)^2=50$ and the line $y=-2x+8$ intersect each other.
5. a) Consider the circle $x^2+y^2=10$. Find an equation for the tangent line drawn to the circle at the point $(3,1)$.
   b) Consider the circle $6x-2y+x^2+y^2=15$. Find an equation for the tangent line drawn to the circle at the point $(1,-2)$.
6. a) Find the points where the circles $(x+4)^2+(y+1)^2=10$ and $(x+1)^2+(y-5)^2=25$ intersect each other.
   b) Find the points where the circles $(x+2)^2+(y+2)^2=50$ and $(x-2)^2+(y-1)^2=25$ intersect each other.

Practice Problems

1. Find an equation for each of the following circles. C will denote the center and r the radius.
   a) $C(6,0)$ $r=7$ b) $C(0,-2)$ $r=2$ c) $C(-8,3)$ $r=\sqrt{3}$
2. Graph the equation $x^2+y^2+14y=4(x+y-1)$
3. Consider the circle $(x+7)^2+(y-3)^2=100$. Find all points on the circle
   a) with $x-$coordinate -1.
   b) with $y-$coordinate 13.
   c) with $x-$coordinate 4.
   d) with $x-$coordinate -2.
4. Find the coordinates of all point(s) where the circle and the line intersect each other.
   a) $(x+5)^2+(y-7)^2=8$ and $x+y=6$
   b) $(x-2)^2+(y+1)^2=50$ and $y=-x+9$
   c) $x^2+(y+4)^2=25$ and $y=x-24$
   d) $(x-3)^2+(y+1)^2=25$ and $y=-x+1$
5. Given the equation of a circle and a point P on it, find an equation for the tangent line drawn to the circle at the point P.
   a) $x^2+y^2=100$ and $P(-8,6)$
   b) $(x-3)^2+(y+3)^2=50$ and $P(10,-4)$
6. Find the coordinates of all points where the given circles intersect each other.
   a) $(x-1)^2+(y-2)^2=10$ and $(x-10)^2+(y-5)^2=40$
   b) $(x-3)^2+y^2=20$ and $x^2+(y-1)^2=50$
   c) $x^2+(y-8)^2=26$ and $(x-7)^2+(y-3)^2=4$

Sample Problems - Answers

1.) $(x-2)^2+(y+1)^2=25$ 2.) $(x+5)^2+(y-3)^2=4$ 3.) $(-3,-5)$ and $(5,-5)$

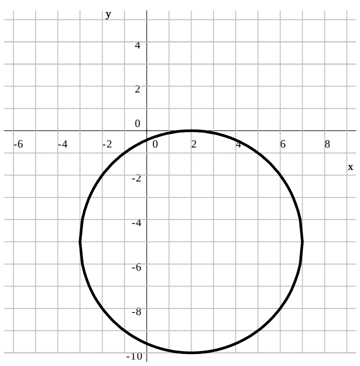
4.) $(-1,10)$ and $(5,-2)$ 5.) a) $y=-3x+10$ b) $y+2=\frac{4}{3}(x-1)$ or $y=\frac{4}{3}x-\frac{10}{3}$

6.) a) $(-1,0)$ and $(-5,2)$ b) $(-1,5)$ and $(5,-3)$

Practice Problems - Answers

1.) a) $(x-6)^2+y^2=49$ b) $x^2+(y+2)^2=4$ c) $(x+8)^2+(y-3)^2=3$

2.) $(x-2)^2+(y+5)^2=25$ center: $(2,-5)$ radius: 5



3.) a) $(-1,11)$ and $(-1,-5)$ b) $(-7,13)$ c) there is no such point

d) $\left(-2,3-5\sqrt{3}\right)$ and $\left(-2,3+5\sqrt{3}\right)$

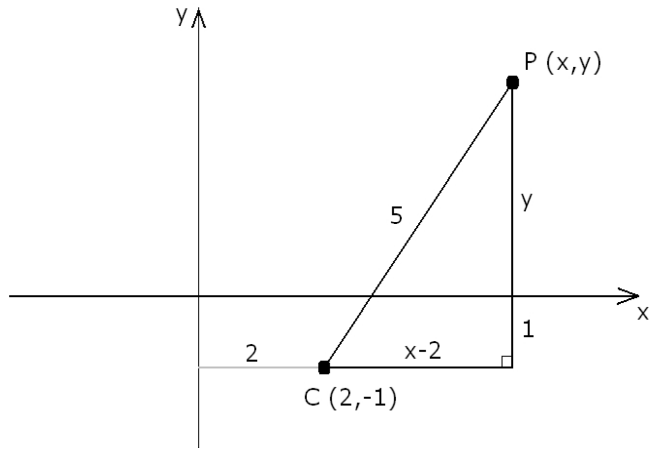
4.) a) $(-3,9)$ b) $(3,6)$ and $(9,0)$ c) they don't intersect d) $(-1,2)$ and $(6,-5)$

5.) a) $\frac{4}{3}(x+8)=y-6$ or $y=\frac{4}{3}x+\frac{50}{3}$ b) $7(x-10)=y+4$ or $y=7x-74$

6.) a) $(4,3)$ b) $(7,2)$ and $(5,-4)$ c) the circles do not intersect

Sample Problems - Solutions

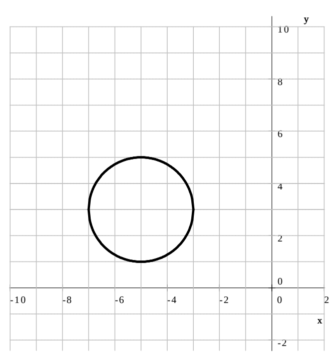
1. Find an equation for the circle centered at $(2,-1)$ with radius $r=5$ units.
Solution: Let $P(x,y)$ be a general point on the circle. If the point P is on the circle, then it must be 5 units away from the point $C(2,-1)$. Consider the right triangle shown on the picture below. The sides of this triangle are $|x-2|$, $|y+1|$, and 5. The Pyhtagorean theorem, stated for this triangle is $(x-2)^2+(y+1)^2=25$, which is the equation for the circle.



2. Graph the equation $10x+x^2+y^2=6(y-5)$
Solution: Complete the squares to read the center and radius.

$$\begin{aligned}
10x+x^2+y^2 &= 6(y-5)\\
10x+x^2+y^2 &= 6y-30\\
x^2+10x+y^2-6y &= -30\\
\underbrace{x^2+10x+25}-25+\underbrace{y^2-6y+9}-9 &= -30\\
(x+5)^2+(y-3)^2-34 &= -30\\
(x+5)^2+(y-3)^2 &= 4
\end{aligned}$$

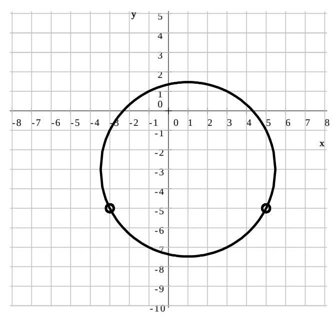
Thus the center is $(-5,3)$ and the radius 2.



3. Consider the circle $(x-1)^2+(y+3)^2=20$. Find all points on the circle with $y-$coordinate being -5.
Solution: We set $y=-5$ and solve for x in the circle's equation. Because the equation is quadratic, we may obtain two or one or no solution. It is also helpful to sketch a graph of the circle; it is centered at $(1,-3)$ and has a radius of $\sqrt{20}$.

$$\begin{aligned}(x-1)^2+(-5+3)^2&=20\\(x-1)^2+(-2)^2&=20\\(x-1)^2+4&=20\\(x-1)^2-16&=0\end{aligned}\qquad\qquad\begin{gathered}(x-1)^2-4^2=0\\(x-1+4)(x-1-4)=0\\(x+3)(x-5)=0\\x_1=-3\qquad x_2=5\end{gathered}$$

Therefore, there are two points on this circle with $y-$coordinate -5, and they are $(-3,-5)$ and $(5,-5)$.



4. Find the coordinates of all points where the circle $x^2+(y-3)^2=50$ and the line $y=-2x+8$ intersect each other.
Solution: We need to solve the following system of equations:

$$\begin{cases}x^2+(y-3)^2=50\\ \quad y=-2x+8\end{cases}$$

We will use substitution. We substitute $y=-2x+8$ in the first equation and solve for x.

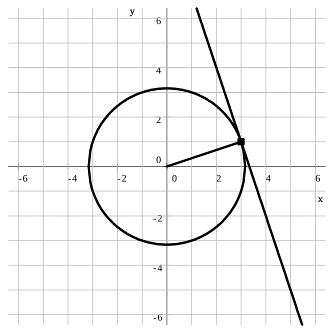
$$\begin{aligned}x^2+(-2x+8-3)^2&=50\\x^2+(-2x+5)^2&=50\\x^2+4x^2-20x+25&=50\\5x^2-20x+25&=50\end{aligned}\qquad\qquad\begin{aligned}&5x^2-20x-25=0\\&5\left(x^2-4x-5\right)=0\\&5(x-5)(x+1)=0\\&x_1=5\qquad x_2=-1\end{aligned}$$

We now use the second equation to find the $y-$value belonging to the $x-$values we just obtained. If $x=5$, then $y=-2(5)+8=-2$. If $x=-1$, then $y=-2(-1)+8=10$. Thus the two points are $(5,-2)$ and $(-1,10)$. We check: both points should be on both the circle and the line.

5. Consider the circle $x^2+y^2=10$. Find an equation for the tangent line drawn to the circle at the point $(3,1)$.
Solution: **The tangent line is perpendicular to the radius drawn to the point of tangency.** We can easily find the slope of this radius as the slope of the line segment connecting the center of the circle, $(0,0)$ and the point of tangency, $(3,1)$. We use the slope formula,

$$\text{slope} = m = \frac{\text{rise}}{\text{run}} = \frac{y_2-y_1}{x_2-x_1} = \frac{1-0}{3-0} = \frac{1}{3}$$

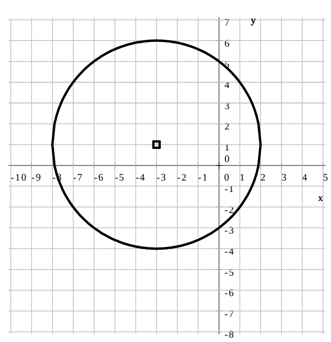
Since perpendicular to a line with slope $\frac{1}{3}$, the tangent line must have slope -3, the negative reciprocal of $\frac{1}{3}$. It must also pass through the point $(3,1)$. The point-slope form of this line's equation is then $y-1=-3(x-3)$. We simplify this and obtain $y=-3x+10$.



b) Consider the circle $6x-2y+x^2+y^2=15$. Find an equation for the tangent line drawn to the circle at the point $(1,-2)$.
Solution: We first transform the equation of the circle to determine its center's coordinates.

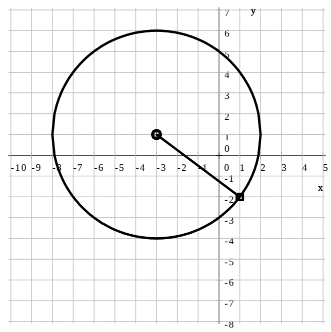
$$\begin{aligned}
6x-2y+x^2+y^2 &= 15\\
x^2+6x+y^2-2y &= 15\\
\underbrace{x^2+6x+9}-9+\underbrace{y^2-2y+1}-1 &= 15\\
(x+3)^2-9+(y-1)^2-1 &= 15\\
(x+3)^2+(y-1)^2 &= 25
\end{aligned}$$

Thus the center of the circle is $(-3,1)$ and its radius is 5.

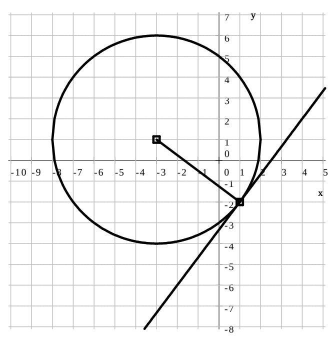


The tangent line is perpendicular to the radius drawn to the point of tangency. We can easily find the slope of this radius: the slope of the line segment connecting the center $(-3, 1)$ and the point of tangency $(1, -2)$ is

$$\text{slope} = m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 1}{1 - (-3)} = -\frac{3}{4}$$



Since perpendicular to a line with slope $-\frac{3}{4}$, the tangent line must have slope $\frac{4}{3}$, the negative reciprocal of $-\frac{3}{4}$. The tangent line must also pass through the point $(1, -2)$. The point-slope form of this line's equation is then $y + 2 = \frac{4}{3}(x - 1)$. We can simplify this and obtain the slope-intercept form, $y = \frac{4}{3}x - \frac{10}{3}$.



6. a) Find the points where the circles $(x+4)^2 + (y+1)^2 = 10$ and $(x+1)^2 + (y-5)^2 = 25$ intersect each other.
Solution: We need to solve the following system:

$$\begin{cases} (x+4)^2 + (y+1)^2 = 10 \\ (x+1)^2 + (y-5)^2 = 25 \end{cases}$$

We multiply out the complete squares and combine like terms in both equations.

$$\begin{aligned} x^2 + 8x + 16 + y^2 + 2y + 1 &= 10 \\ x^2 + 8x + y^2 + 2y + 17 &= 10 \\ x^2 + 8x + y^2 + 2y &= -7 \end{aligned} \qquad \begin{aligned} x^2 + 2x + 1 + y^2 - 10y + 25 = 25 \\ x^2 + 2x + y^2 - 10y + 26 = 25 \\ x^2 + 2x + y^2 - 10y = -1 \end{aligned}$$

We will multiply the second equation by -1 and add the two equations. This will cancel out all quadratic terms.

$$\begin{aligned} x^2+8x+y^2+2y &= -7 \\ -x^2-2x-y^2+10y &= 1 \end{aligned}$$

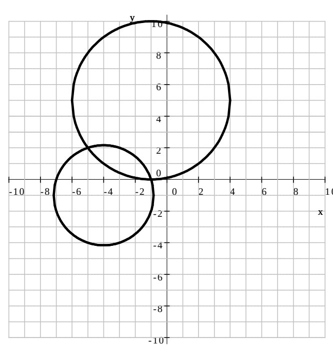
The sum of the two equations is

$$\begin{aligned} 6x+12y &= -6 \quad \text{divide both sides by 6} \\ x+2y &= -1 \\ x &= -2y-1 \end{aligned}$$

We substitute this into the first equation and solve for y.

$$\begin{aligned} (x+4)^2+(y+1)^2 &= 10 \quad \text{and} \quad x=-2y-1 \\ (x+4)^2+(y+1)^2 &= 10 \\ \left(\underbrace{-2y-1}_{x}+4\right)^2+(y+1)^2 &= 10 \\ (-2y+3)^2+(y+1)^2 &= 10 \\ 4y^2-12y+9+y^2+2y+1 &= 10 \\ 5y^2-10y+10 &= 10 \\ 5y^2-10y &= 0 \\ 5y\,(y-2) &= 0 \qquad \Longrightarrow \quad y_1=0 \qquad y_2=2 \end{aligned}$$

We now find the x values belonging to the $y-$values, using $x=-2y-1$. If $y_1=0$, then $x_1=-2\cdot 0-1=-1$ and if $y_2=2$, then $x_2=-2\cdot 2-1=-5$. Thus the two circles intersect at the points $(-1,0)$ and $(-5,2)$.



b) Find the points where the circles $(x+2)^2+(y+2)^2=50$ and $(x-2)^2+(y-1)^2=25$ intersect each other.
Solution: We need to solve the following system:

$$\begin{cases} (x+2)^2+(y+2)^2=50 \\ (x-2)^2+(y-1)^2=25 \end{cases}$$

We multiply out the complete squares and combine like terms in both equations

$$\begin{aligned} x^2+4x+4+y^2+4y+4 &= 50 \\ x^2+4x+y^2+4y &= 42 \end{aligned} \qquad \begin{aligned} x^2-4x+4+y^2-2y+1=25 \\ x^2-4x+y^2-2y=20 \end{aligned}$$

We will multiply the second equation by -1 and add the two equations. This will cancel out all quadratic terms.

$$\begin{aligned} x^2+4x+y^2+4y &= 42 \\ -x^2+4x-y^2+2y &= -20 \end{aligned}$$

The sum of the two equations is

$$\begin{aligned} 8x+6y &= 22 \quad \text{divide both sides by 2} \\ 4x+3y &= 11 \end{aligned}$$

We now solve for y

$$y=\frac{-4x+11}{3}=-\frac{4}{3}x+\frac{11}{3}$$

We substitute this into the first equation and solve for x.

$$\begin{aligned}
(x+2)^2+(y+2)^2 &= 50 \qquad \text{and} \qquad y=-\frac{4}{3}x+\frac{11}{3} \\
(x+2)^2+\left(\underbrace{-\frac{4}{3}x+\frac{11}{3}}_{y}+2\right)^2 &= 50 \qquad\qquad \frac{11}{3}+2=\frac{17}{3} \\
(x+2)^2+\left(\frac{-4x+17}{3}\right)^2 &= 50 \\
(x+2)^2+\frac{(-4x+17)^2}{3^2} &= 50 \qquad \text{since } \left(\frac{a}{b}\right)^2=\frac{a^2}{b^2} \\
(x+2)^2+\frac{(-4x+17)^2}{9} &= 50 \qquad \text{multiply by 9} \\
9(x+2)^2+(-4x+17)^2 &= 450 \\
9\left(x^2+4x+4\right)+16x^2-136x+289 &= 450 \\
9x^2+36x+36+16x^2-136x+289 &= 450 \\
25x^2-100x-125 &= 0 \\
25\left(x^2-4x-5\right) &= 0 \\
25(x-5)(x+1) &= 0 \qquad \Longrightarrow \quad x_1=5 \qquad x_2=-1
\end{aligned}$$

We now find the y values belonging to the $x-$values, using $y=\dfrac{-4x+11}{3}$. If $x=5$, then $y=\dfrac{-4(5)+11}{3}=-3$ and if $x=-1$, then $y=\dfrac{-4(-1)+11}{3}=5$. Thus the two circles intersect at the points $(-1,5)$ and $(5,-3)$.

For more documents like this, visit our page at http://www.teaching.martahidegkuti.com and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.

