In what follows, we will study the equation of circles.

**Definition:** A **circle** is the set of all points in the plane equidistant to a fixed point. That distance is called the **radius** of the circle, and that fixed point is called the **center** of the circle.

Recall the distance formula, an application of the Pythagorean Theorem.

The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ can be computed as

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Consider now the circle centered at $C(3,1)$, with radius 5. The general point, $P(x,y)$ will be on this circle if and only if its distance from $C$ is exactly 5 units. We express this using the distance formula with $P(x,y)$ and $C(3,1)$.

$$\begin{aligned} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} &= d \\ \sqrt{(x-3)^2 + (y-1)^2} &= 5 \qquad \text{square both sides} \\ (x-3)^2 + (y-1)^2 &= 25 \end{aligned}$$

This last statement is the equation of the circle.

The equation of the circle centered at $C(k,h)$ with radius $r > 0$ is

$$(x-k)^2 + (y-h)^2 = r^2$$

**Example 1.** Find an equation for the circle centered at $(2,-7)$ with radius $r = 3$ units.

**Solution:** We apply the distance formula between $P(x,y)$, the general point on the circle, and $C(2,-7)$, the center of the circle. The point $P(x,y)$ is on the circle if and only if it is exactly 3 units away from $C(2,-7)$. Be careful to substract $-7$ and not 7.

$$\begin{aligned} \sqrt{(x-2)^2 + (y-(-7))^2} &= 3 \qquad \text{square both sides} \\ (x-2)^2 + (y+7)^2 &= 9 \end{aligned}$$

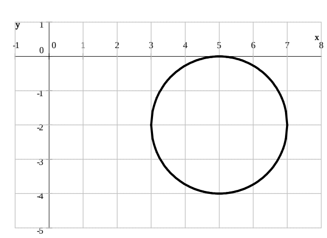
So the equation of this circle is $(x-2)^2 + (y+7)^2 = 9$.

**Example 2.** In each case, state the center and radius of the circle given its equation. Sketch the graph of the circle.

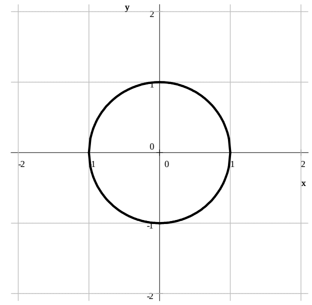
a) $(x-5)^2 + (y+2)^2 = 4$ b) $x^2 + y^2 = 1$ c) $2x + x^2 + y^2 + 8 = 8y$

**Solution:** a) If the circle has equation $(x-k)^2 + (y-h)^2 = r^2$ is centered at $(k,h)$, then $k = 5$ and $h = -2$.

The expression $y + 2$ can (and should) be interpreted here as $y - (-2)$. Thus the center of the circle is $C(5,-2)$, and its radius is 2.

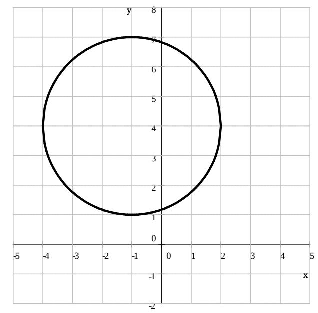


b) The equation $x^2+y^2=1$ can be interpreted as $(x-0)^2+(y-0)^2=1$. Therefore, its center is the origin, and it has radius 1. This circle is called **the unit circle** and will be important in future courses.



c) The equation $2x+x^2+y^2+8=8y$ is not in the form that easily allows us to read the center and radius. Therefore, we first need to bring it to that form. We will reduce one side to zero and complete the square (twice!) to obtain the standard form.

$$\begin{aligned}
2x+x^2+y^2+8 &= 8y && \text{subtract } 8y\\
x^2+2x+y^2-8y+8 &= 0\\
\underbrace{x^2+2x+1}-1+\underbrace{y^2-8y+16}-16+8 &= 0\\
(x+1)^2+(y-4)^2-9 &= 0 && \text{add } 9\\
(x+1)^2+(y-4)^2 &= 9
\end{aligned}$$



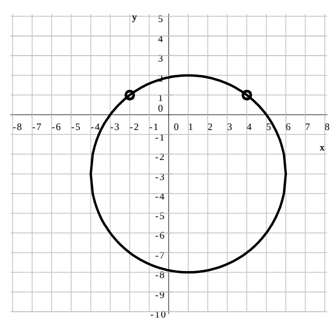
Now we can easily tell that this circle is centered at $(-1,4)$, and has radius 3.

**Example 3.** Consider the circle $(x-1)^2+(y+3)^2=25$. Find all points on the circle with

a) $y-$coordinate 1 b) $x-$coordinate $-4$ c) $y-$coordinate 3

**Solution:** a) We set $y=1$ and solve for $x$ in the circle's equation. Because the equation is quadratic, we may obtain two or one or no solution. It is also helpful to sketch a graph of the circle; it is centered at $(1,-3)$ and its radius is 5 units long.

$$\begin{aligned}
(x-1)^2+(1+3)^2 &= 25\\
(x-1)^2+4^2 &= 25\\
(x-1)^2+16 &= 25\\
(x-1)^2-9 &= 0\\
(x-1)^2-3^2 &= 0\\
(x-1+3)(x-1-3) &= 0\\
(x+2)(x-4) &= 0\\
x_1 &= -2 \quad x_2=4
\end{aligned}$$

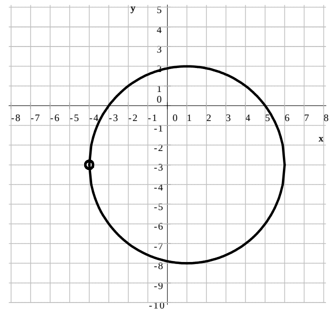


Therefore, there are two points on this circle with $y-$coordinate 1, and they are $\boxed{(-2,1) \text{ and } (4,1)}$.

b) We set $x=-4$ and solve for $y$ in the circle's equation. Because the equation is quadratic, we may obtain two or one or no solution.

$$(-4-1)^2+(y+3)^2=25$$

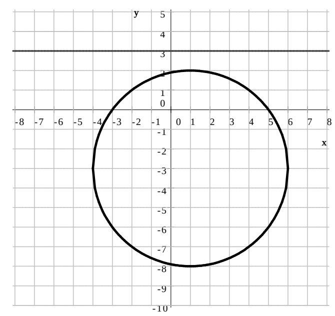
$$
\begin{aligned}
(-4-1)^2+(y+3)^2 &= 25\\
(-5)^2+(y+3)^2 &= 25\\
25+(y+3)^2 &= 25\\
(y+3)^2 &= 0\\
y &= -3
\end{aligned}
$$



Therefore, there is only one point on this circle with $x-$coordinate $-4$, and it is $\boxed{(-4,-3)}$.

c) We set $y=3$ and solve for $x$ in the circle's equation. Because the equation is quadratic, we may obtain two or one or no solution.

$$
\begin{aligned}
(x-1)^2+(3+3)^2 &= 25\\
(x-1)^2+6^2 &= 25\\
(x-1)^2+36 &= 25\\
(x-1)^2+11 &= 0
\end{aligned}
$$



This equation has no real solution, indicating that there is $\boxed{\text{no point}}$ with $y-$coordinate 3 on this circle.

These problems can be interpreted as looking for the intersection of a circle and a (horizontal or vertical) line. The lines however do not always need to be horizontal or vertical. To find the intersection of a circle and a line, we solve the system of equation formed by the two equations.

**Example 4.** Find both coordinates of all points where the circle $(x-2)^2+(y-1)^2=25$ and line $-x+3y=6$ intersect each other.

**Solution:** To find where these intersect, we will solve a system of equation.

$$
\begin{cases}
(x-2)^2+(y-1)^2=25\\
-x+3y=6
\end{cases}
$$

The second equation is linear in both in $x$ and $y$. Because $y$ has there a coefficient 3 and $x$ a coefficient $-1$, we will solve for $x$ and substitute that in the equation of the circle.

We solve for $x$. $\quad x=3y-6$ We substitute this into the first equation.

$(x-2)^2+(y-1)^2=25$ becomes $\quad (3y-6-2)^2+(y-1)^2=25.$ We solve for $y$.

$$
\begin{aligned}
(3y-8)^2+(y-1)^2 &= 25\\
9y^2-48y+64+y^2-2y+1 &= 25 \quad \text{combine like terms}\\
10y^2-50y+65 &= 25 \quad \text{subtract 25}\\
10y^2-50y+40 &= 0 \quad \text{factor}\\
10\left(y^2-5y+4\right) &= 0\\
10(y-1)(y-4) &= 0\\
y_1=1 \quad y_2 &= 4
\end{aligned}
$$

The two solutions for $y$ indicate that there are two intersection points. We go back to the expression we substituted to find $x$ in both cases.

$$x = 3y - 6 \quad \Longrightarrow \quad x_1 = 3 \cdot 1 - 6 = -3 \text{ and } x_2 = 3 \cdot 4 - 6 = 6$$

Thus, the intersection points are $(-3, 1)$ and $(6, 4)$. We check: these points must be on both the circle and the line. Therefore, the coordinates of both points must be solutions of the original system of equation.

Check $(-3, 1)$:

$$\text{LHS} = (-3-2)^2 + (1-1)^2 = (-5)^2 + 0^2 = 25 = \text{RHS} \ \checkmark \quad \text{and}$$

$$\text{LHS} = -(-3) + 3 \cdot 1 = 3 + 3 = 6 = \text{RHS} \ \checkmark$$

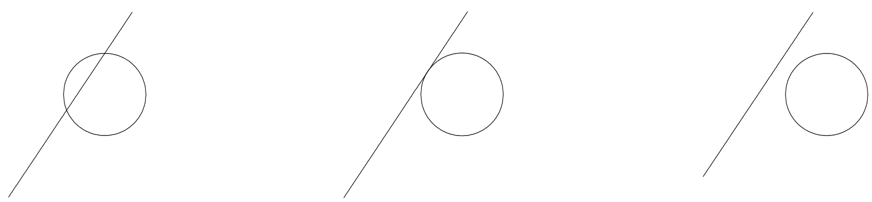
Check $(6, 4)$:

$$\text{LHS} = (4-2)^2 + (4-1)^2 = 4^2 + 3^2 = 16 + 9 = 25 = \text{RHS} \ \checkmark \quad \text{and}$$

$$\text{LHS} = -6 + 3 \cdot 4 = -6 + 12 = 6 = \text{RHS} \ \checkmark$$

Thus our solution, $\boxed{(-3, 1) \text{ and } (6, 4)}$, is correct.

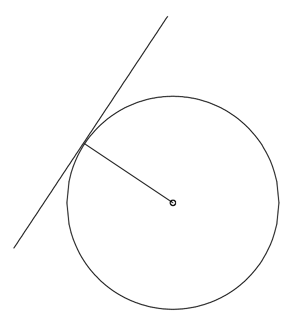
After the substitution, we end up with a quadratic equation. Such an equation can have two solutions, one solution, or no real solution. A circle and a line in the same plane can have two intersection points, one intersection point, or none. This conclusion agrees with the possibilities we find using a geometric approach.



Definition: If a circle and a line in the same plane have exactly one intersection point, we say that the line is **tangent** to the circle.

Tangent lines are an important concept and will be subject of further study in calculus. In case of circles, the tangent line has a useful property.

**Theorem:** The radius drawn to the point of tangency is perpendicular to the tangent line.



This property will enable us to find tangent lines to circles.

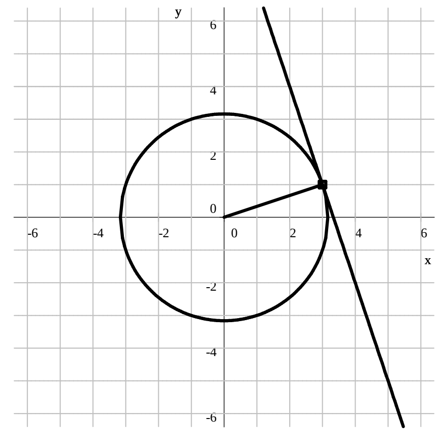
 

**Example 5.** Consider the circle $x^2+y^2=10$. Find an equation for the tangent line drawn to the circle at the point $(3,1)$.

**Solution:** Recall that **the tangent line is perpendicular to the radius drawn to the point of tangency.** We can easily find the slope of this radius as the slope of the line segment connecting the center of the circle, $(0,0)$ and the point of tangency, $(3,1)$. We use the slope formula.

$$\text{slope} = m = \frac{\text{rise}}{\text{run}} = \frac{y_2-y_1}{x_2-x_1} = \frac{1-0}{3-0} = \frac{1}{3}$$

Since perpendicular to a line with slope $\frac{1}{3}$, the tangent line must have slope $-3$, the negative reciprocal of $\frac{1}{3}$. It must also pass through the point $(3,1)$. The point-slope form of this line's equation is then $y-1=-3(x-3)$. We simplify this and obtain $y=-3x+10$. So the tangent line is $\boxed{y=-3x+10}$.



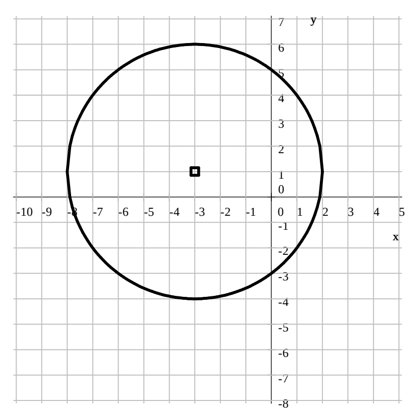

How could we check our result? If we look for the intersection of the line and the circle, the quadratic equation we obtain will have only one solution.

**Example 6.** Consider the circle $6x-2y+x^2+y^2=15$. Find an equation for the tangent line drawn to the circle at the point $(1,-2)$.

**Solution:** We first transform the equation of the circle to determine its center's coordinates.

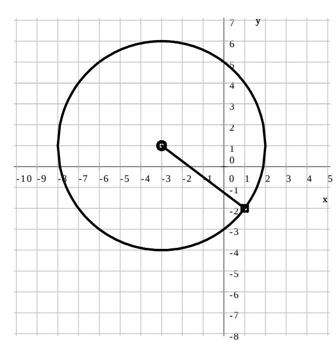
$$\begin{aligned}
6x-2y+x^2+y^2 &= 15\\
x^2+6x+y^2-2y &= 15\\
\underbrace{x^2+6x+9}-9+\underbrace{y^2-2y+1}-1 &= 15\\
(x+3)^2-9+(y-1)^2-1 &= 15\\
(x+3)^2+(y-1)^2 &= 25
\end{aligned}$$

Thus the center of the circle is $(-3,1)$ and its radius is 5.

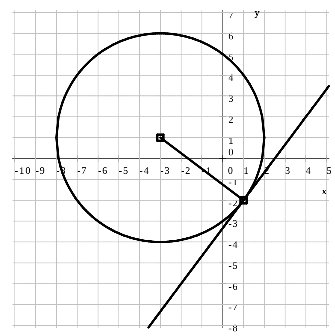



**The tangent line is perpendicular to the radius drawn to the point of tangency.** We can easily find the slope of this radius: the slope of the line segment connecting the center $(-3,1)$ and the point of tangency $(1,-2)$ is

$$\text{slope} = m = \frac{\text{rise}}{\text{run}} = \frac{y_2-y_1}{x_2-x_1} = \frac{-2-1}{1-(-3)} = -\frac{3}{4}$$



Since perpendicular to a line with slope $-\dfrac{3}{4}$, the tangent line must have slope $\dfrac{4}{3}$, the negative reciprocal of $-\dfrac{3}{4}$. The tangent line must also pass through the point $(1,-2)$. The point-slope form of this line's equation is then $y+2=\dfrac{4}{3}(x-1)$. We can simplify this and obtain the slope-intercept form, $y=\dfrac{4}{3}x-\dfrac{10}{3}$.



We can also find the intersection of two circles. But in the case of two circles, both equations in the system are quadratic in both $x$ and $y$, making it difficult to solve for one in terms of the other. We get around this problem by first subtracting the two equations. When we do that, both $x^2$ and $y^2$ will be cancelled out.

**Example 7.** Find both coordinates of all points where the circles $(x+5)^2+(y-3)^2=25$ and $(x+2)^2+(y+3)^2=10$ intersect each other.

**Solution:** We expand the complete squares and combine like terms in both equations.

$$\begin{aligned}(x+5)^2+(y-3)^2&=25\\ x^2+10x+25+y^2-6y+9&=25\\ x^2+10x+y^2-6y+9&=0\\ x^2+10x+y^2-6y&=-9\end{aligned}\qquad\qquad\begin{aligned}(x+2)^2+(y+3)^2&=10\\ x^2+4x+4+y^2+6y+9&=10\\ x^2+4x+y^2+6y+13&=10\\ x^2+4x+y^2+6y&=-3\end{aligned}$$

To subtract is to add the opposite. Instead of subtracting, we will multiply one equation by $-1$ and then add.

$$\begin{array}{rl} -x^2-10x-y^2+6y=9 & \\ +\quad x^2\ +4x\ +y^2+6y=-3 & \\ \hline -6x+12y=6 & \text{divide by } 6\\ -x+2y=1 & \text{solve for } x\\ 2y-1=x & \end{array}$$

Now that we have a linear equation in $x$ and $y$, we can solve for one in terms of the other and then substitute into either of the equation. We have that $\boxed{x=2y-1}$. We substitute this into the second equation.

$(x+5)^2+(y-3)^2=25$ becomes

$$\begin{aligned}(2y-1+5)^2+(y-3)^2&=25. && \text{We solve for } y.\\ (2y+4)^2+(y-3)^2&=25\\ 4y^2+16y+16+y^2-6y+9&=25 && \text{combine like terms}\\ 5y^2+10y+25&=25 && \text{subtract 25}\\ 5y^2+10y&=0 && \text{factor}\\ 5y(y+2)&=0\\ y_1=0\quad y_2&=-2\end{aligned}$$

The two solutions for $y$ indicate that there are two intersection points. We go back to the expression we substituted to find $x$ in both cases.

$$x=2y-1\quad\Longrightarrow\quad x_1=2\cdot 0-1=-1 \text{ and } x_2=2(-2)-1=-5$$

Thus, the intersection points are $\boxed{(-1,0) \text{ and } (-5,-2)}$. We check: these points must be on both circles. The coordinates of both points must be solutions of the original system of equation. We leave checking for the reader.

## Practice Problems

1. Find an equation for each of the following circles. $C$ will denote the center and $r$ the radius.

   a) $C\,(6,0)\quad r=7$ b) $C\,(0,-2)\quad r=2$ c) $C\,(-8,3)\quad r=\sqrt{3}$

2. Graph the equation $x^2+y^2+14y=4\,(x+y-1)$

3. Consider the circle $(x+7)^2+(y-3)^2=100$. Find all points on the circle

   a) with $x-$coordinate $-1$.
   b) with $y-$coordinate $13$.
   c) with $x-$coordinate $4$.
   d) with $x-$coordinate $-2$.

4. Find the coordinates of all points where the given line and circle intersect each other.

   a) $(x-3)^2+(y+2)^2=10$ and $x+2y=4$
   b) $(x+1)^2+(y-4)^2=18$ and $y=x+11$
   c) $(x+8)^2+(y+1)^2=60$ and $y=2x-5$

5. Given the equation of a circle and a point $P$ on it, find an equation for the tangent line drawn to the circle at the point $P$.

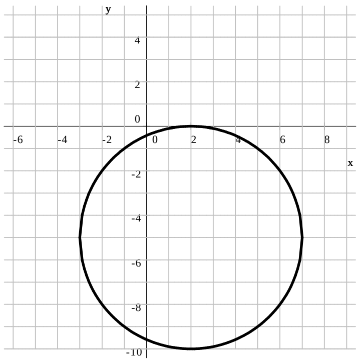
   a) $x^2+y^2=100$ and $P\,(-8,6)$ b) $(x-3)^2+(y+3)^2=50$ and $P\,(10,-4)$

6. Find the coordinates of all points where the given circles intersect each other.

   a) $(x+1)^2+(y-2)^2=10$ and $(x-1)^2+(y-6)^2=50$
   b) $(x+5)^2+(y-3)^2=13$ and $(x-4)^2+(y+3)^2=52$
   c) $(x+5)^2+(y-1)^2=60$ and $(x+7)^2+(y-2)^2=105$

# Answers

1. a) $(x-6)^2+y^2=49$ b) $x^2+(y+2)^2=4$ c) $(x+8)^2+(y-3)^2=3$
2. $(x-2)^2+(y+5)^2=25$ center: $(2,-5)$ radius: 5




3. a) $(-1,11)$ and $(-1,-5)$ b) $(-7,13)$ c) there is no such point d) $\left(-2,3-5\sqrt{3}\right)$ and $\left(-2,3+5\sqrt{3}\right)$
4. a) $(6,-1)$ and $(2,1)$ b) $(-4,7)$ c) no intersection point
5. a) $\frac{4}{3}(x+8)=y-6$ or $y=\frac{4}{3}x+\frac{50}{3}$ b) $7(x-10)=y+4$ or $y=7x-74$
6. a) $(0,-1)$ and $(-4,1)$ b) $(-2,1)$ c) no intersection point

