In what follows, we will study the equation of circles.

Definition: A **circle** is the set of all points in the plane equidistant to a fixed point. That distance is called the **radius** of the circle, and that fixed point is called the **center** of the circle.

Recall the distance formula, an application of the Pythagorean Theorem.

The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ can be computed as

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Consider now the circle centered at $C(3,1)$, with radius 5. The general point, $P(x,y)$ will be on this circle if and only if its distance from C is exactly 5 units. We express this using the distance formula with $P(x,y)$ and $C(3,1)$.

$$\begin{aligned}
\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} &= d \\
\sqrt{(x-3)^2 + (y-1)^2} &= 5 \qquad \text{square both sides} \\
(x-3)^2 + (y-1)^2 &= 25
\end{aligned}$$

This last statement is the equation of the circle.

The equation of the circle centered at $C(k,h)$ with radius $r > 0$ is

$$(x-k)^2 + (y-h)^2 = r^2$$

Example 1. Find an equation for the circle centered at $(2,-7)$ with radius $r = 3$ units.

Solution: We apply the distance formula between $P(x,y)$, the general point on the circle, and $C(2,-7)$, the center of the circle. The point $P(x,y)$ is on the circle if and only if it is exactly 3 units away from $C(2,-7)$. Be careful to substract -7 and not 7.

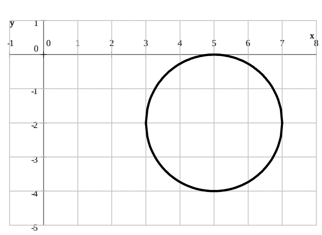
$$\begin{aligned}
\sqrt{(x-2)^2 + (y-(-7))^2} &= 3 \qquad \text{square both sides} \\
(x-2)^2 + (y+7)^2 &= 9
\end{aligned}$$

So the equation of this circle is $(x-2)^2 + (y+7)^2 = 9$.

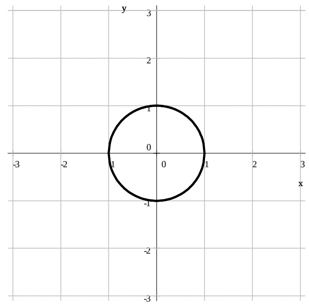
Example 2. In each case, state the center and radius of the circle given its equation. Sketch the graph of the circle.

a) $(x-5)^2 + (y+2)^2 = 4$ b) $x^2 + y^2 = 1$ c) $2x + x^2 + y^2 + 8 = 8y$

Solution: a) If the circle has equation $(x-k)^2 + (y-h)^2 = r^2$ is centered at (k,h), then $k = 5$ and $h = -2$. The expression $y+2$ can (and should) be interpreted here as $y-(-2)$. Thus the center of the circle is $C(5,-2)$, and its radius is 2.



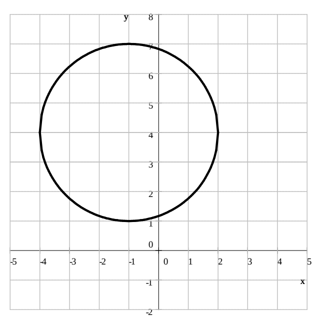
b) The equation $x^2 + y^2 = 1$ can be interpreted as $(x-0)^2 + (y-0)^2 = 1$. Therefore, its center is the origin, and it has radius 1. Such a circle is called **the unit circle** and will be very important in future courses such as trigonometry.



c) The equation $2x + x^2 + y^2 + 8 = 8y$ is not in the form that easily allows us to read the center and radius. Therefore, we first need to bring it to that form. We will reduce one side to zero and complete the square (twice!) to obtain the standard form.

$$\begin{aligned} 2x + x^2 + y^2 + 8 &= 8y \qquad \text{subtract } 8y \\ x^2 + 2x + y^2 - 8y + 8 &= 0 \\ \underbrace{x^2 + 2x + 1} - 1 + \underbrace{y^2 - 8y + 16} - 16 + 8 &= 0 \\ (x+1)^2 + (y-4)^2 - 9 &= 0 \qquad \text{add } 9 \\ (x+1)^2 + (y-4)^2 &= 9 \end{aligned}$$

Now we can easily tell that this circle is centered at $(-1, 4)$, and has radius 3.



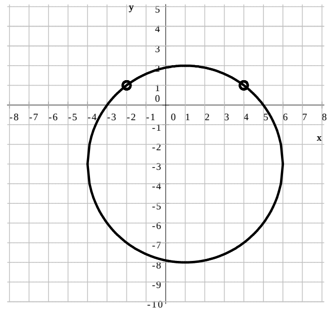
Example 3. Consider the circle $(x-1)^2 + (y+3)^2 = 25$. Find all points on the circle with

a) $y-$coordinate 1 b) $x-$coordinate -4 c) $y-$coordinate 3

Solution: a) We set $y = 1$ and solve for x in the circle's equation. Because the equation is quadratic, we may obtain two or one or no solution. It is also helpful to sketch a graph of the circle; it is centered at $(1, -3)$ and has radius 5 units long.

$$\begin{aligned} (x-1)^2 + (1+3)^2 &= 25 \\ (x-1)^2 + 4^2 &= 25 \\ (x-1)^2 + 16 &= 25 \\ (x-1)^2 - 9 &= 0 \end{aligned} \qquad \begin{aligned} (x-1)^2 - 3^2 &= 0 \\ (x-1+3)(x-1-3) &= 0 \\ (x+2)(x-4) &= 0 \\ x_1 = -2 \quad x_2 &= 4 \end{aligned}$$

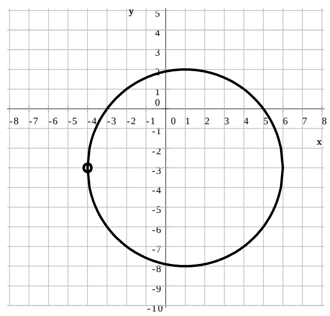
Therefore, there are two points on this circle with $y-$coordinate 1, and they are $(-2, 1)$ and $(4, 1)$.



b) We set $x = -4$ and solve for y in the circle's equation. Because the equation is quadratic, we may obtain two or one or no solution.

$$\begin{aligned}
(-4-1)^2 + (y+3)^2 &= 25 \\
(-5)^2 + (y+3)^2 &= 25 \\
25 + (y+3)^2 &= 25 \\
(y+3)^2 &= 0 \\
y &= -3
\end{aligned}$$

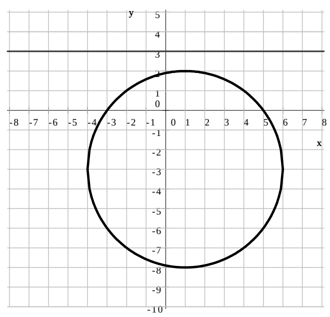
Therefore, there is only one point on this circle with $x-$coordinate -4, and it is $(-4,-3)$.



c) We set $y = 3$ and solve for x in the circle's equation. Because the equation is quadratic, we may obtain two or one or no solution.

$$\begin{aligned}
(x-1)^2 + (3+3)^2 &= 25 \\
(x-1)^2 + 6^2 &= 25 \\
(x-1)^2 + 36 &= 25 \\
(x-1)^2 + 11 &= 0
\end{aligned}$$

This equation has no real solution, indicating that the circle does not contain a point with $y-$coordinate 3.

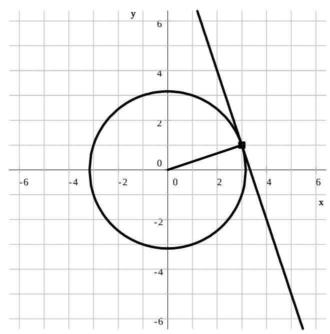


Example 4. Consider the circle $x^2 + y^2 = 10$. Find an equation for the tangent line drawn to the circle at the point $(3, 1)$.

Solution: Recall that **the tangent line is perpendicular to the radius drawn to the point of tangency.** We can easily find the slope of this radius as the slope of the line segment connecting the center of the circle, $(0, 0)$ and the point of tangency, $(3, 1)$. We use the slope formula.

$$\text{slope} = m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{3 - 0} = \frac{1}{3}$$

Since perpendicular to a line with slope $\frac{1}{3}$, the tangent line must have slope -3, the negative reciprocal of $\frac{1}{3}$. It must also pass through the point $(3, 1)$. The point-slope form of this line's equation is then $y - 1 = -3\,(x - 3)$. We simplify this and obtain $y = -3x + 10$. So the tangent line is $y = -3x + 10$.

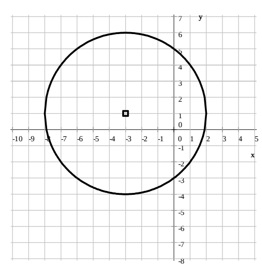


Example 5. Consider the circle $6x - 2y + x^2 + y^2 = 15$. Find an equation for the tangent line drawn to the circle at the point $(1, -2)$.

Solution: We first transform the equation of the circle to determine its center's coordinates.

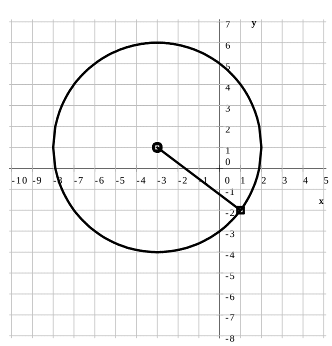
$$\begin{aligned}
6x - 2y + x^2 + y^2 &= 15 \\
x^2 + 6x + y^2 - 2y &= 15 \\
\underbrace{x^2 + 6x + 9} - 9 + \underbrace{y^2 - 2y + 1} - 1 &= 15 \\
(x+3)^2 - 9 + (y-1)^2 - 1 &= 15 \\
(x+3)^2 + (y-1)^2 &= 25
\end{aligned}$$

Thus the center of the circle is $(-3, 1)$ and its radius is 5.

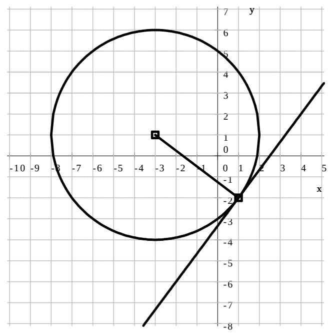


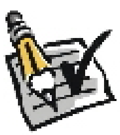
The tangent line is perpendicular to the radius drawn to the point of tangency. We can easily find the slope of this radius: the slope of the line segment connecting the center $(-3, 1)$ and the point of tangency $(1, -2)$ is

$$\text{slope} = m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 1}{1 - (-3)} = -\frac{3}{4}$$



Since perpendicular to a line with slope $-\frac{3}{4}$, the tangent line must have slope $\frac{4}{3}$, the negative reciprocal of $-\frac{3}{4}$. The tangent line must also pass through the point $(1,-2)$. The point-slope form of this line's equation is then $y+2=\frac{4}{3}(x-1)$. We can simplify this and obtain the slope-intercept form, $y=\frac{4}{3}x-\frac{10}{3}$.





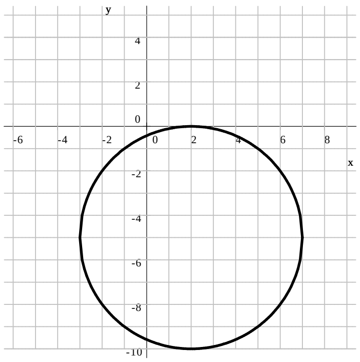
Practice Problems

1. Find an equation for each of the following circles. C will denote the center and r the radius.
   a) $C(6,0)$ $r=7$ b) $C(0,-2)$ $r=2$ c) $C(-8,3)$ $r=\sqrt{3}$
2. Graph the equation $x^2+y^2+14y=4(x+y-1)$
3. Consider the circle $(x+7)^2+(y-3)^2=100$. Find all points on the circle
   a) with $x-$coordinate -1. c) with $x-$coordinate 4.
   b) with $y-$coordinate 13. d) with $x-$coordinate -2.
4. Given the equation of a circle and a point P on it, find an equation for the tangent line drawn to the circle at the point P.
   a) $x^2+y^2=100$ and $P(-8,6)$ b) $(x-3)^2+(y+3)^2=50$ and $P(10,-4)$

Answers

1. a) $(x-6)^2+y^2=49$ b) $x^2+(y+2)^2=4$ c) $(x+8)^2+(y-3)^2=3$

2. $(x-2)^2+(y+5)^2=25$ center: $(2,-5)$ radius: 5



3. a) $(-1,11)$ and $(-1,-5)$ b) $(-7,13)$ c) there is no such point d) $\left(-2,3-5\sqrt{3}\right)$ and $\left(-2,3+5\sqrt{3}\right)$

4. a) $\dfrac{4}{3}(x+8)=y-6$ or $y=\dfrac{4}{3}x+\dfrac{50}{3}$ b) $7(x-10)=y+4$ or $y=7x-74$

