Sample Problems

1. Consider the circles $x^2+y^2=36$ and $(x-5)^2+y^2=16$. Let C be the intersection of the two common tangent lines drawn to the circle.

   a) Find the coordinates of C.

   b) Consider one of the common tangent lines. Find the distance between the points of tangency.

2. Let C_1 and C_2 be circles defined by $(x-10)^2+y^2=36$ and $(x+15)^2+y^2=81$, respectively. Consider one of the common tangent lines. Find the distance between the points of tangency.

3. Given are the points $A(-3,6)$ and $B(0,2)$. Consider the circle $x^2+y^2-16x-4y+43=0$. Find the tangent lines that are parallel with line segment AB.

4. Find an equation of the circle that passes through the points $A(0,2)$, $B(-2,-2)$, and $C(-8,-4)$.

Practice Problems

1. Consider the circles $(x+5)^2+y^2=100$ and $x^2+y^2=49$. Let C be the intersection of the two common tangent lines drawn to the circle.

   a) Find the coordinates of C.

   b) Consider one of the common tangent lines. Find the distance between the points of tangency.

2. Let C_1 and C_2 be circles defined by $x^2+(y+7)^2=1$ and $x^2+(y-6)^2=16$, respectively. Consider one of the common tangent lines. Find the distance between the points of tangency.

3. Given are the points $A(8,-1)$ and $B(12,1)$. Consider the circle $10x-2y+x^2+y^2+6=0$. Find the tangent lines that are parallel with line segment AB.

4. Given the three points, find an equation of the circle that contains all three points.

   a) $P(0,0)$, $Q(0,8)$, and $R(-6,0)$.

   b) $A(-1,6)$, $B(-7,12)$, and $C(-3,0)$.

   c) $X(3,-7)$, $Y(5,-1)$, and $Z(-3,-5)$.

Sample Problems - Answers

1.) a) $(15, 0)$ b) $\sqrt{21}$ 2.) $\sqrt{616} = 2\sqrt{154}$ or 20 3.) $y = -\dfrac{4}{3}x + \dfrac{13}{3}$ and $y = -\dfrac{4}{3}x + 21$

4.) $(x + 7)^2 + (y - 3)^2 = 50$

Practice Problems - Answers

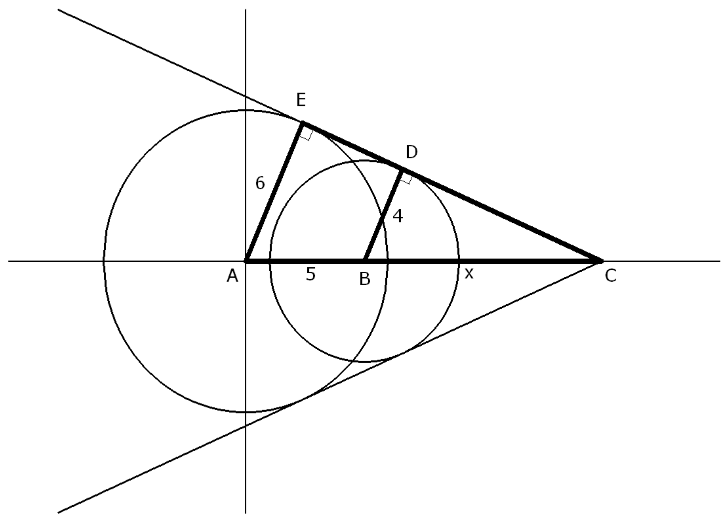
1.) a) $\left(\dfrac{35}{3}, 0\right)$ b) 4 2.) $4\sqrt{10}$ or 12 3.) $y = \dfrac{1}{2}x - \dfrac{3}{2}$ and $y = \dfrac{1}{2}x + \dfrac{17}{2}$

4.) a) $(x + 3)^2 + (y - 4)^2 = 25$ b) $(x + 8)^2 + (y - 5)^2 = 50$ c) $(x - 1)^2 + (y + 3)^2 = 20$

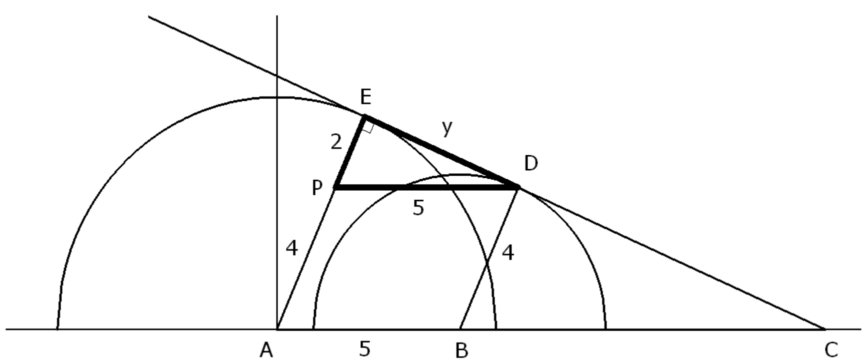
Sample Problems - Solutions

1. Consider the circles $x^2 + y^2 = 36$ and $(x-5)^2 + y^2 = 16$. Let C be the intersection of the two common tangent lines drawn to the circle.

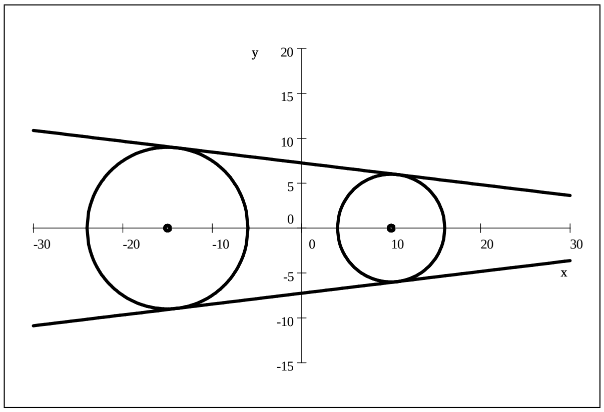
   a) Find the coordinates of C.
   Solution: Using the notation shown on the picture below, triangles ACE and BCD are similar. Based on that, we write the equation $\dfrac{6}{4} = \dfrac{x+5}{x}$. The solution of this equation is 10. Thus C is $(15, 0)$.

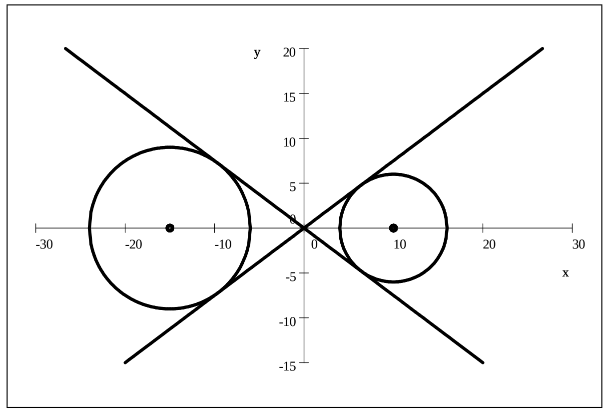


   b) Consider one of the common tangent lines. Find the distance between the points of tangency.
   Solution: We draw a horizontal line through the point D. Let us denote the intersection of AE and the horizontal line by P. The quadrilateral $ABDP$ is a parallelogram since opposite sides are parallel. Consequently, opposite sides are equally long. This means that $AP = 4$ and so $PE = 2$. Also, $PD = 5$. We can now compute y via the Pythagorean theorem: $y = \sqrt{21}$.



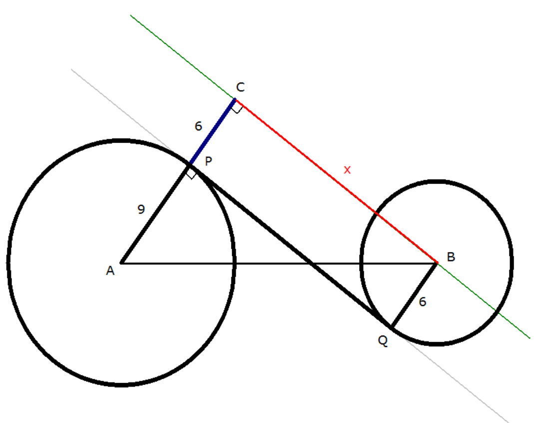
2. Let C_1 and C_2 be circles defined by $(x-10)^2+y^2=36$ and $(x+15)^2+y^2=81$, respectively. Consider one of the common tangent lines. Find the distance between the points of tangency.
Solution: This problem is different from the previous one because these two circles do not intersect each other. Because of that, there are not two but four common tangent lines to these circles.



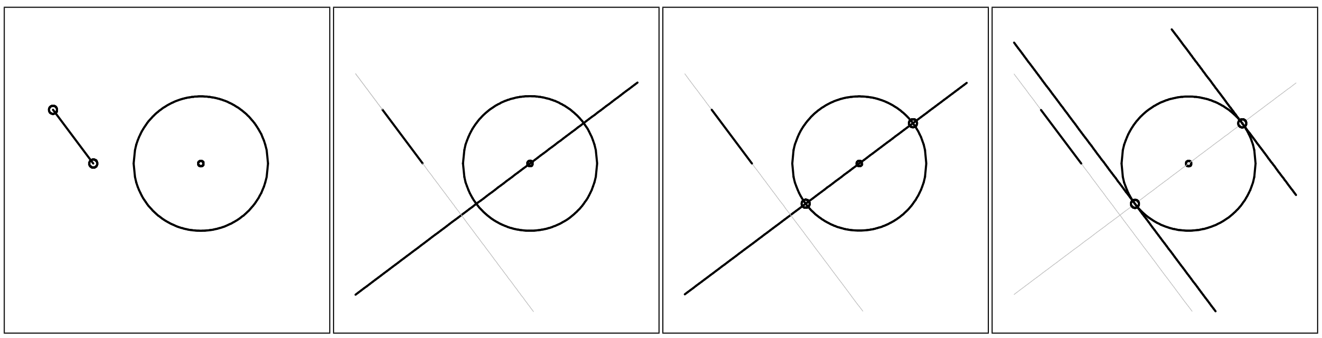


In case of the first picture, the problem is identical to the previous problem. We compute the distance between the points of tangency using the same steps as shown above. The distance is $\sqrt{616}=2\sqrt{154}$. We now turn our attention to the second picture.

From the equations of the circles we conclude that the centers are at $(10,0)$ and $(-15,0)$, and the radii are 6 and 9 units. The two circles are then 25 units apart.
Let A and B denote the centers of the two circles. We draw the radius $\overline{AP}$ beyond the point P. We draw a line that passes through B and is parallel to $\overline{PQ}$. Let C denote the point of intersection.
Since $\overline{PQ}$ is a line tangent to both circles, the radii $\overline{BQ}$ and $\overline{AP}$ are perpendicular to $\overline{PQ}$. Thus $PQBC$ is a rectangle. Consider now the triangle ABC. There is a right angle at C, the hypotenuse ($\overline{AB}$) is 25 units long, the leg $\overline{AC}$ is 15 units long. Thus the other leg, $\overline{BC}$ is 20 units long, by the Pythagorean Theorem. Since $PQBC$ is a rectangle, $\overline{BC}=\overline{PQ}=20$.



3. Given are the points $A(-3,6)$ and $B(0,2)$. Consider the circle $x^2+y^2-16x-4y+43=0$. Find the tangent lines that are parallel with line segment AB.
Solution: To solve this problem, we will use the property of circles that **the radius drawn to the point of tangency is perpendicular to the tangent line**.
Step 1. We determine the center of the circle and the slope of the line AB.
Step 2. We come up with the equation of a line k that passes through the center of the circle and is perpendicular to line segment AB.
Step 3. We find all points where the circle and line k intersect. Since **the tangent line is perpendicular to the radius drawn to the point of tangency,** these points are the points of tangency.
Step 4. We come up with the equations of the tangent lines. Their slope is the slope of line segment AB and they pass through the points found in step 3.



Step 1 Step 2 Step 3 Step 4

Step 1. The circle is

$$\begin{aligned}
x^2+y^2-16x-4y+43 &= 0\\
x^2-16x+y^2-4y+43 &= 0\\
x^2-16x+64-64+y^2-4y+4-4+43 &= 0\\
(x-8)^2+(y-2)^2 &= 25
\end{aligned}$$

Thus the center is $(8,2)$ and the radius is 5. We compute the slope of line segment AB using the slope formula.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{2-6}{0-(-3)}=-\frac{4}{3}$$

Step 2. The tangent line and the radius drawn to the point of tangency are perpendicular: thus we want a straight line that is perpendicular to the line segment AB and passes through the center. This means a slope of $\frac{3}{4}$, and passing through $(8,2)$. The point-slope form of this line is $y-2=\frac{3}{4}(x-8)$.

Step 3. We intersect this with the circle to find the points of tangency, i.e. solve the system

$$\begin{cases} (x-8)^2 + (y-2)^2 = 25 \\ y - 2 = \dfrac{3}{4}(x-8) \end{cases}$$

We will use substitution. Given the equations, it is a smart trick to substitute $y-2$ instead of y.

$$\begin{aligned} (x-8)^2 + \left(\frac{3}{4}(x-8)\right)^2 &= 25 \\ (x-8)^2 + \frac{9}{16}(x-8)^2 &= 25 \qquad 1 + \frac{9}{16} = \frac{25}{16} \\ \frac{25}{16}(x-8)^2 &= 25 \qquad \text{divide by } 25 \\ \frac{1}{16}(x-8)^2 &= 1 \qquad \text{multiply by } 16 \\ (x-8)^2 &= 16 \qquad \text{subtract } 16 \end{aligned}$$

$$\begin{aligned} (x-8)^2 - 16 &= 0 \\ (x-8+4)(x-8-4) &= 0 \\ (x-4)(x-12) &= 0 \\ x_1 = 4 \qquad x_2 &= 12 \end{aligned}$$

We find the $y-$values using the equation $y - 2 = \dfrac{3}{4}(x-8)$

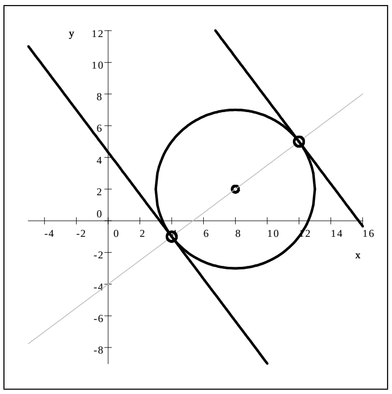
$$\begin{aligned} y_1 &= \frac{3}{4}(x_1 - 8) + 2 = \frac{3}{4}(4-8) + 2 = -3 + 2 = -1 \quad \text{and} \\ y_2 &= \frac{3}{4}(x_2 - 8) + 2 = \frac{3}{4}(12-8) + 2 = 3 + 2 = 5 \end{aligned}$$

Thus the points of tangency are $(4, -1)$ and $(12, 5)$.

Step 4. The slope of both tangent lines must be the same as that of line segment AB. Thus the tangent lines:

$$\begin{aligned} m &= -\frac{4}{3} \quad T_1 = (4, -1) \quad \Longrightarrow \quad y + 1 = -\frac{4}{3}(x-4) \\ m &= -\frac{4}{3} \quad T_1 = (12, 5) \quad \Longrightarrow \quad y - 5 = -\frac{4}{3}(x-12) \end{aligned}$$

After simplifying these equations we get $y = -\dfrac{4}{3}x + \dfrac{13}{3}$ and $y = -\dfrac{4}{3}x + 21$



4. Find an equation of the circle that passes through the points $A(0,2)$, $B(-2,-2)$, and $C(-8,-4)$.

We will present two different solutions.

Solution 1. This solution is based on the following fact. The **perpendicular bisector of a line segment** AB is the set of all points in the plane that are equidistant to points A and B. Recall that the center of the circle is equidistant to all points on the circle. Recall that the line segment connecting two points on a circle is call a chord. The center of a circle is contained on the perpendicular bisector of any of its chords.

Step 1. Let us find the equation of the perpendicular bisector of line segment AB. It is a line that is perpendicular to AB and passes through its midpoint, M.

$$m_{AB} = \frac{y_A - y_B}{x_A - x_B} = \frac{2-(-2)}{0-(-2)} = \frac{4}{2} = 2 \quad \Longrightarrow \quad m_{\text{perpendicular}} = -\frac{1}{2}$$

and the midpoint M can be found as

$$x_M = \frac{x_A + x_B}{2} = \frac{0+(-2)}{2} = -1 \quad \text{and} \quad y_M = \frac{y_A + y_B}{2} = \frac{2+(-2)}{2} = 0 \qquad M_{AB}(-1,0)$$

Thus the perpendicular bisector of AB is a line with slope $-\frac{1}{2}$, passing through $(-1,0)$. We can easily find this equation:

$$y = -\frac{1}{2}(x+1) = -\frac{1}{2}x - \frac{1}{2}$$

Step 2. We repeat the entire procedure with line segment AC.

$$m_{AC} = \frac{y_A - y_C}{x_A - x_C} = \frac{2-(-4)}{0-(-8)} = \frac{6}{8} = \frac{3}{4} \quad \Longrightarrow \quad m_{\text{perpendicular}} = -\frac{4}{3}$$

and the midpoint M can be found as

$$x_M = \frac{x_A + x_C}{2} = \frac{0+(-8)}{2} = -4 \quad \text{and} \quad y_M = \frac{y_A + y_C}{2} = \frac{2+(-4)}{2} = -1 \qquad M_{AC}(-4,-1)$$

Thus the perpendicular bisector of AC is a line with slope $-\frac{4}{3}$, passing through $(-4,-1)$. We can easily find this equation:

$$\begin{aligned} y+1 &= -\frac{4}{3}(x+4) \\ y &= -\frac{4}{3}(x+4) - 1 = -\frac{4}{3}x - \frac{16}{3} - 1 = -\frac{4}{3}x - \frac{19}{3} \end{aligned}$$

Step 3. The center of the circle lies on both perpendicular bisectors found in Steps 1 and 2. Thus it must be the intersection of those two lines. We find the intersection point by solving the system

$$\begin{aligned} y &= -\frac{1}{2}x - \frac{1}{2} \\ y &= -\frac{4}{3}x - \frac{19}{3} \end{aligned}$$

We can easily use substitution and obtain the equation

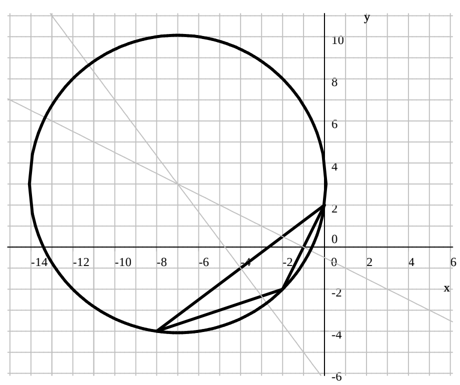
$$\begin{aligned} -\frac{1}{2}x - \frac{1}{2} &= -\frac{4}{3}x - \frac{19}{3} \qquad \text{multiply by 6} \\ -3x - 3 &= -8x - 38 \\ 5x - 3 &= -38 \\ 5x &= -35 \\ x &= -7 \qquad y = -\frac{1}{2}x - \frac{1}{2} = -\frac{1}{2}(-7) - \frac{1}{2} = \frac{7}{2} - \frac{1}{2} = 3 \end{aligned}$$

and obtain $(-7, 3)$.

Step 4. To find the radius, we need to compute the distance between $(-7, 3)$ and any of the points A, B, and C. Let us compute the distance of $(-7, 3)$ from A.

$$r = \sqrt{(-7 - x_A)^2 + (3 - y_A)^2} = \sqrt{(-7 - 0)^2 + (3 - 2)^2} = \sqrt{50}$$

Thus the equation of the circle is $(x + 7)^2 + (y - 3)^2 = 50$.



Solution 2. Suppose that the cicle has an equation

$$(x-k)^2+(y-h)^2=r^2$$

To find this equation, we must find the real numbers k, h, and r^2. Since there are three unknown variables, we will need three equations. We obtain those by stating that points A, B, and C are all on the circle.

$$\begin{array}{rrcll}
1) & (0-k)^2+(2-h)^2 & = & r^2 & A(0,2) \\
2) & (-2-k)^2+(-2-h)^2 & = & r^2 & B(-2,-2) \\
3) & (-8-k)^2+(-4-h)^2 & = & r^2 & C(-8,-4)
\end{array}$$

We simplify the equations

$$\begin{array}{rrclcrl}
1) & k^2+h^2-4h+4 & = & r^2 & \quad & 1) & k^2+h^2-4h+4=r^2 \\
2) & k^2+4k+4+h^2+4h+4 & = & r^2 & & 2) & k^2+4k+h^2+4h+8=r^2 \\
3) & k^2+16k+64+h^2+8h+16 & = & r^2 & & 3) & k^2+16k+h^2+8h+80=r^2
\end{array}$$

We can eliminate r and all expressions quadratic in k if we subtract the first equation from the second and the third equations. We will proceed carefully, adding the opposite instead of subtracting.

$$\begin{array}{rrcl}
1) & -k^2-h^2+4h-4 & = & -r^2 \\
2) & k^2+4k+h^2+4h+8 & = & r^2 \\
 & & \Downarrow & \\
 & 4k+8h+4 & = & 0 \\
 & k+2h & = & -1
\end{array}
\qquad
\begin{array}{rr}
1) & -k^2-h^2+4h-4=-r^2 \\
3) & k^2+16k+h^2+8h+80=r^2 \\
 & \Downarrow \\
 & 16k+12h+76=0 \\
 & 4k+3h=-19
\end{array}$$

We obtained a system in two variables. Also, this system is now completely linear.

$$\begin{array}{rclccr}
k+2h & = & -1 & \text{multiply by } -4 & \Longrightarrow & -4k-8h=4 \\
4k+3h & = & -19 & & & 4k+3h=-19
\end{array}$$

We add the two equations and obtain $-5h=-15$ and so $h=3$. From the equation $k+2h=-1$ we easily obtain the value $k=-7$. We now substitute $k=-7$, $h=3$ into the first equation to obtain the value of r^2.

$$\begin{array}{rcll}
k^2+(2-h)^2 & = & r^2 & \qquad k=-7,\ h=3 \\
(-7)^2+(2-3)^2 & = & r^2 & \\
50 & = & r^2 &
\end{array}$$

And so the equation of the circle is $(x+7)^2+(y-3)^2=50$. It is interesting how the two solutions presented here relate to each other.

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